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Three-Wave Quasi-Kinetic Approximation in the Problem of the Evolution of a Spectrum of Nonlinear Gravity Waves at Small Depths

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Abstract—A three-wave kinetic equation is generalized for a statistical description of nonlinear waves with weak dispersion, which forbids three-wave resonances, for example, of surface gravity waves at small depths in the Boussinesq approximation. This approach is based on the derivation of a kinetic equation by using the effective rate of wave attenuation $\beta(\mathbf{k})$ due to nonlinear interactions. However, unlike the case of an exact three-wave frequency resonance, in our final results, the passage to the limit $\beta \longrightarrow 0$ is not performed, and spread δ-functions of the form $\delta_\beta(\sigma(k) \pm \sigma(k_1) \pm \sigma(k_2))$ with spreading parameter $\beta(\mathbf{k})$ are retained in place of δ-functions in frequency $\delta(\sigma(k) \pm \sigma(k_1) \pm \sigma(k_2))$. An additional equation can be obtained to determine $\beta(\mathbf{k})$. As a result, we arrive at a closed problem for the wave spectrum $N(\mathbf{k})$ that evolves due to three-wave interactions of weakly dispersive waves. The final system of equations for $N(\mathbf{k})$ and $\beta(\mathbf{k})$ differs from the common three-wave kinetic equation and is conventionally referred to as a quasi-kinetic equation.

1. INTRODUCTION

The problem of describing the evolution of waves in water is largely a problem of describing their nonlinear interactions. In the context of statistical theory, this problem has been studied for more than 30 years starting with the well-known work of Hasselmann [1], where the following four-wave kinetic equation for a wave-action spectrum $N(\mathbf{k})$ was derived for gravity waves at finite depth h in a horizontally homogeneous case:

$$\begin{aligned}\frac{\partial N(\mathbf{k})}{\partial t} = P^0(\mathbf{k}) = 4\pi\iiint d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \\ \times T^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\{N(\mathbf{k}_2)N(\mathbf{k}_3)[N(\mathbf{k}) + N(\mathbf{k}_1)] \\ - N(\mathbf{k})N(\mathbf{k}_1)[N(\mathbf{k}_2) + N(\mathbf{k}_3)]\}\delta(\sigma(k) + \sigma(k_1) \\ - \sigma(k_2) - \sigma(k_3))\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3).\end{aligned} \tag{1.1}$$

Here, $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is a combination of matrix elements contributing to four-wave interactions (see, for example, [2]) and $\sigma(k)$ is the dispersion function for gravity waves in a water of finite depth h:

$$\sigma(k) = [gk\tanh(kh)]^{1/2}, \tag{1.2}$$

where k is the magnitude of the wave vector $\mathbf{k} = (k_x, k_y)$ and g is the acceleration of gravity. In (1.1) and below, time dependence in $N(\mathbf{k})$ is omitted for brevity.

Most studies devoted to the kinetic equation (1.1) and its applications to predicting wind waves relate to a deep-water case, when the dispersion relation (1.2) takes the form

$$\sigma(k) = (gk)^{1/2}. \tag{1.3}$$

However, there are a fairly large number of studies in which the four-wave kinetic equation (1.1) is used at finite depths, i.e., for $k_m h \le 1$ and even $k_m h \ll 1$, where k_m is the wave number of a peak in a wave spectrum $N(\mathbf{k})$. As examples, we refer to studies [3–6], where the integral of nonlinear interactions $P(\mathbf{k})$ was calculated numerically at small $k_m h$, and the features of nonlinear energy transfer at finite depths were compared to those in a deep-water case.

Actually, as shown in [7], all the above results of considering the four-wave kinetic equation (1.1) for $k_m h < 0.5$–1 are unjustified. The problem is that deriving equation (1.1) uses a special procedure of filtering quadratic terms of the initial dynamic equations, which are associated with nonresonance three-wave interactions [2, 8]. This procedure is based on the assumption that the contribution of these interactions to the dynamics of a wave spectrum is small compared to that of resonance four-wave interactions. In a deep-water case, this assumption is actually satisfied when the nonlinear parameter ε takes on the values that are typical of real wind waves [9]. However, similar calculations for $k_m h \le 1$ show that, as the depth decreases, nonresonance three-wave interactions increase significantly in importance, and, even at $kh \simeq 0.8$, the usual technique of deriving the four-wave kinetic equation (1.1) becomes inapplicable.

Therefore, in the case of finite-depth waves when $kh \leq 1$, the usual four-wave kinetic equation (1.1) must be replaced by an equation that takes into account three-wave interactions. An attempt at deriving a three-wave kinetic equation seems to be reasonable. However, it is well known that the conditions of three-wave resonances of the form

$$\sigma(k) \pm \sigma(k_1) \pm \sigma(k_2) = 0, \quad \mathbf{k} \pm \mathbf{k}_1 \pm \mathbf{k}_2 = 0 \quad (1.4)$$

are not satisfied for surface gravity waves with the dispersion relation (1.2). The sole exception is the limiting shallow-water case, when it follows from (1.2) in the limit $kh \longrightarrow 0$ that

$$\sigma(k) = (gh)^{1/2}k. \quad (1.5)$$

Then relations (1.4) are identically satisfied for collinear vectors $\mathbf{k}$, $\mathbf{k}_1$, and $\mathbf{k}_2$. Waves obeying relation (1.5) are referred to as semidispersive waves. It is for this dispersive relation that a three-wave kinetic equation was obtained in [10].

However, this kinetic equation differs significantly from the usual kinetic equation, because the time scale for such interactions turns out to be proportional to $t^{3/2}$; i.e., the collision integral is proportional to $t^{1/2}$. In addition, its applicability at small but finite kh remains open to question. Nevertheless, with some modifications, this equation was used in [11] to analyze the spectrum of shallow-water gravity waves.

On the other hand, if $kh < 1$, the general dispersion relation (1.2) can be written in the Boussinesq approximation in the form

$$\sigma(k) = (gh)^{1/2}k\left[1 - \frac{1}{6}(kh)^2\right], \quad (1.6)$$

which is obtained by expanding the hyperbolic tangent in a power series in kh. Although the resonance conditions of three-wave interactions (1.4) are not fulfilled for the dispersion relation (1.6) as before, a small mismatch

$$\Delta\sigma = \Delta\sigma(k, \pm k_1, \pm k_2) \equiv \sigma(k) \pm \sigma(k_1) \pm \sigma(k_2) \quad (1.7)$$

of the frequency resonance (1.4) occurs for a narrow wave spectrum:

$$\gamma = \frac{\Delta\sigma}{\sigma(k)} = \frac{1}{6}\left[(kh)^2 \mp \frac{k_1}{k}(k_1h)^2 \mp \frac{k_2}{k}(k_2h)^2\right] \ll 1. \quad (1.8)$$

Within some limits, this result allows the use of the conventional technique of deriving a three-wave kinetic equation in the case of random waves with the dispersion relation (1.6).

Off-resonance interactions are described by using the effective rate $\beta(\mathbf{k})$ of wave attenuation due to nonlinear interactions [12]. As distinguished from the case of an exact frequency resonance, passing to the limit $\beta \longrightarrow 0$ is not performed in our final results. As a result, spread δ-functions, which are characterized by the spreading parameter $\beta(\mathbf{k})$, are retained in the collision integral $P^0(N)$ in place of frequency δ-functions. An additional equation can be obtained to determine $\beta(\mathbf{k})$. In the Boussinesq approximation (1.6), we arrive at a closed problem of describing a gravity-wave spectrum $N(\mathbf{k})$ that evolves due to three-wave interactions.

The system of joint equations finally obtained for $N(\mathbf{k}, \beta)$ and $\beta(\mathbf{k}, N)$ is conventionally called the quasi-kinetic approximation. This approximation is also suitable for describing weakly nonlinear random waves with a weak nondecaying dispersion of any other physical origin (see [13] for examples of weakly dispersive nonlinear waves).

2. INITIAL RELATIONS

We will consider a potential model of surface gravity waves in a fluid layer $\eta > z > -h = \text{const}$, where $\eta = \eta(\mathbf{x}, t)$ is a perturbation of the free boundary, $\mathbf{x} = (x, y)$ are the horizontal coordinates, and z is the upward-directed vertical coordinate. The corresponding system of dynamic equations for the velocity potential $\varphi(\mathbf{x}, z, t)$ and the elevation level $\eta(\mathbf{x}, t)$ is simplified after taking the Fourier transforms with respect to the horizontal coordinates $\mathbf{x}$ of the form

$$\varphi(\mathbf{x}, z, t) = (1/2\pi)\int\hat{\varphi}(\mathbf{k}, z, t)\exp(i\mathbf{k}\mathbf{x})d\mathbf{k}, \quad (2.1a)$$

$$\eta(\mathbf{x}, t) = (1/2\pi)\int\hat{\eta}(\mathbf{k}, t)\exp(i\mathbf{k}\mathbf{x})d\mathbf{k}. \quad (2.1b)$$

In a weakly nonlinear case ($\varepsilon = k\eta \ll 1$), one can obtain an approximate system of two equations for $\hat{\eta}(\mathbf{k}, t)$ and the velocity potential $\hat{\varphi}(\mathbf{k}, z, t)$ at the surface $\hat{\eta}(\mathbf{k}, t)$ from the equations for $\hat{\varphi}_s(\mathbf{k}, t)$ and $\eta(\mathbf{x}, t)$. This system for $\hat{\eta}(\mathbf{k}, t)$ and $\hat{\varphi}_s(\mathbf{k}, t)$ is simplified in turn by introducing the "normal" complex variables

$$a(\mathbf{k}, t) = \left(\frac{\sigma(k)}{2g}\right)^{1/2}\hat{\eta}(\mathbf{k}, t) + i\left(\frac{g}{2\sigma(k)}\right)^{1/2}\hat{\varphi}_s(\mathbf{k}, t), \quad (2.2)$$

where $\sigma(k)$ is given by relation (1.2). The desired variable $\eta(\mathbf{x}, t)$ is expressed through $a(\mathbf{k}, t)$ in the following way:

$$\eta(\mathbf{x}, t) = \frac{1}{2\pi}\int d\mathbf{k}\left(\frac{g}{2\sigma(k)}\right)^{1/2}[a(\mathbf{k}, t)\exp(i\mathbf{k}\mathbf{x}) + a^*(\mathbf{k}, t)\exp(-i\mathbf{k}\mathbf{x})]. \quad (2.3)$$

In the normal variables, the system of equations for two variables $\hat{\eta}(\mathbf{k}, t)$ and $\hat{\varphi}_s(\mathbf{k}, t)$ reduces to one complex equation for $a(\mathbf{k}, t)$:

$$\frac{\partial a(\mathbf{k})}{\partial t} + i\sigma(k)a(\mathbf{k}) = -i\int d\mathbf{k}_1\int d\mathbf{k}_2 V_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$$

$$\times a(\mathbf{k}_1)a(\mathbf{k}_2)\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) - i\int d\mathbf{k}_1\int d\mathbf{k}_2$$

$$\times V_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a^*(\mathbf{k}_1) a(\mathbf{k}_2) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) \quad (2.4)$$

$$- i \int d\mathbf{k}_1 \int d\mathbf{k}_2 V_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a^*(\mathbf{k}_1)$$

$$\times a^*(\mathbf{k}_2) \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) - \dots .$$

Henceforth, time dependence of the normal variables is omitted. Dots on the right-hand side of (2.4) denote cubic and higher-order terms in $a(\mathbf{k})$, which arise from consideration of higher-order terms of ε-expansions of the initial equations and which are unnecessary in what follows. This procedure of deriving equation (2.4) was proposed by Zakharov in [8], where it was presented at greater length.

The three-wave interaction coefficients V_1, V_2, and V_3 appearing in (2.4) are given in a number of studies for the case of a finite depth. To be specific, we will present these coefficients in the form of [14]:

$$V_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = -V(-\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) - V(-\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1) + V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}), \quad (2.5)$$

$$V_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = 2V_1(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}), \quad (2.6)$$

$$V_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) + V(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1) + V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}), \quad (2.7)$$

where

$$V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{1}{8\pi}\left[\frac{\sigma(k_2)g}{2\sigma(k)\sigma(k_1)}\right]^{1/2} \times [(\mathbf{k}\mathbf{k}_1) + q(k)q(k_1)], \quad (2.8)$$

$$q(k) = k\tanh(kh).$$

The following symmetry conditions are satisfied:

$$V_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = V_1(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1),$$

$$V_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = V_2(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2), \quad (2.9)$$

$$V_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = V_3(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1) = V_3(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2).$$

A statistical description of gravity waves within the framework of the quadratically nonlinear dynamic equation (2.4) uses the spectrum of normal variables $N(\mathbf{k})$. In a horizontally homogeneous case, it is determined by the relation

$$N(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}') = \langle a(\mathbf{k}) a^*(\mathbf{k}') \rangle, \quad (2.10)$$

where the dependence of N on t is not written out explicitly.

By relation (2.3), the so-called nonsymmetric spatial wave spectrum is expressed through $N(\mathbf{k})$:

$$F(\mathbf{k}) = \frac{\sigma(k)}{4\pi^2 g} N(\mathbf{k}), \quad (2.11)$$

which satisfies the normalization condition

$$\int F(\mathbf{k}) d\mathbf{k} = \langle \eta^2 \rangle, \quad (2.12)$$

and which does not possess symmetry in $\mathbf{k}$: $F(\mathbf{k}) \neq F(-\mathbf{k})$. The usual symmetric spatial wave spectrum $\Psi(\mathbf{k})$ is expressed in turn through $F(\mathbf{k})$:

$$\Psi(\mathbf{k}) = \frac{1}{2}[F(\mathbf{k}) + F(-\mathbf{k})], \quad (2.13)$$

The spectrum $\Psi(\mathbf{k}) = \Psi(-\mathbf{k})$ is defined by the standard relations

$$\Psi(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}') = \langle \bar{\eta}(\mathbf{k}, t) \bar{\eta}^*(\mathbf{k}', t) \rangle, \quad (2.14)$$

$$\vec{\eta}(\mathbf{k}, t) = \frac{1}{(2\pi)^2} \int \eta(\mathbf{x}, t) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x} \quad (2.15)$$

and satisfies a normalization condition similar to (2.12). We notice that the Fourier transforms (2.1) and (2.15) for $\eta(\mathbf{x}, t)$ differ in normalization, which causes the factor $(4\pi^2)^{-1}$ to appear in (2.11).

3. DERIVING AN EVOLUTIONARY EQUATION FOR A SPECTRUM

The standard procedure will be used to derive the equation for $N(\mathbf{k})$ from (2.4). We multiply equation (2.4) by $a^*(\mathbf{k}')$ and average the result over an ensemble. We take the conjugate of equation (2.4) with the variable $\mathbf{k}$ replaced by $\mathbf{k}'$, multiply it by $a(\mathbf{k})$, and also average. Adding the equations so obtained and integrating the result with respect to $\mathbf{k}'$, we obtain

$$\frac{\partial N(\mathbf{k})}{\partial t} = -i \int d\mathbf{k}' \int d\mathbf{k}_1 \int d\mathbf{k}_2 \mathrm{Im}[R(\mathbf{k}, \mathbf{k}', \mathbf{k}_1, \mathbf{k}_2)]. \quad (3.1)$$

Here,

$$\begin{aligned} R(\mathbf{k}, \mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) &= V_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \langle a(\mathbf{k}_1) a(\mathbf{k}_2) a^*(\mathbf{k}') \rangle \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \\ &- V_1(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) \langle a^*(\mathbf{k}_1) a^*(\mathbf{k}_2) a(\mathbf{k}) \rangle \delta(\mathbf{k}' - \mathbf{k}_1 - \mathbf{k}_2) \\ &+ V_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \langle a^*(\mathbf{k}_1) a(\mathbf{k}_2) a^*(\mathbf{k}') \rangle \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) \\ &- V_2(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) \langle a(\mathbf{k}_1) a^*(\mathbf{k}_2) a^*(\mathbf{k}) \rangle \delta(\mathbf{k}' + \mathbf{k}_1 - \mathbf{k}_2) \\ &+ V_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \langle a^*(\mathbf{k}_1) a^*(\mathbf{k}_2) a^*(\mathbf{k}') \rangle \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) \\ &- V_3(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) \langle a(\mathbf{k}_1) a(\mathbf{k}_2) a(\mathbf{k}) \rangle \delta(\mathbf{k}' + \mathbf{k}_1 + \mathbf{k}_2). \end{aligned} \quad (3.2)$$

To calculate the third spectral moments on the right-hand side of (3.2), we apply the above procedure. For example, the equation for the moment $\langle a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}')\rangle$ takes the form

$$\frac{\partial}{\partial t}\langle a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}')\rangle + i[\sigma(k_1)+\sigma(k_2)-\sigma(k')] \times \langle a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}')\rangle = T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'), \quad (3.3)$$

where

$$\begin{aligned} T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}') &= -i\int d\mathbf{k}'\int d\mathbf{k}_2'[-V_1(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2) \\ &\times \langle a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}_1')a^*(\mathbf{k}_2')\rangle\delta(\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2') \\ &+ V_2(\mathbf{k}_1, \mathbf{k}_1', \mathbf{k}_2')\langle a^*(\mathbf{k}_1')a(\mathbf{k}_2')a(\mathbf{k}_2)a^*(\mathbf{k}')\rangle \\ &\times\delta(\mathbf{k}_1+\mathbf{k}_1'-\mathbf{k}_2') + V_2(\mathbf{k}_2, \mathbf{k}_1', \mathbf{k}_2') \\ &\times\langle a^*(\mathbf{k}_1')a(\mathbf{k}_2')a(\mathbf{k}_1)a^*(\mathbf{k}')\rangle\delta(\mathbf{k}_2+\mathbf{k}_1'-\mathbf{k}_2') + \ldots]. \end{aligned} \quad (3.4)$$

Dots on the right-hand side of (3.4) denote omitted terms with the fourth moments of a and a^* that involve an unequal number of the multipliers a and a^*, for example, $\langle aaaa^*\rangle$ etc.

The right-hand sides of equations for the third moments of type (3.3) can be expressed through the desired wave spectrum $N(\mathbf{k})$ by the assumption that the spectral components $a(\mathbf{k})$ are approximately Gaussian. This assumption makes it possible to neglect fourth-order cumulants in the expressions for the fourth spectral moments. As a result, only the fourth moments with an equal number of the multipliers a and a^* turn out to be other than zero. In view of (2.10), these moments are expressed through $N(\mathbf{k})$ as follows:

$$\begin{aligned} &\langle a(\mathbf{k}_i)a(\mathbf{k}_j)a^*(\mathbf{k}_l)a^*(\mathbf{k}_m)\rangle \simeq N(\mathbf{k}_i)N(\mathbf{k}_j) \\ &\times[\delta(\mathbf{k}_i-\mathbf{k}_l)\delta(\mathbf{k}_j-\mathbf{k}_m)+\delta(\mathbf{k}_j-\mathbf{k}_l)\delta(\mathbf{k}_i-\mathbf{k}_m)]. \end{aligned} \quad (3.5)$$

Relation (3.5) represents the conventional closure hypothesis that is used in deriving the three-wave kinetic equation. We notice that, when a four-wave kinetic equation of type (1.1) is being derived, the fourth-order cumulants must be taken into account and a similar representation for the sixth-order moments must be used in place of (3.5).

The closure hypothesis (3.5) will not be specially discussed in this work, although its use is more questionable than in a four-wave case. Representation (3.5) can be regarded as the well-known Millionshchikov hypothesis, which is used to close equations for third moments in the theory of hydrodynamic turbulence. However, even a qualitative justification of the closure hypothesis (3.5) requires that the third moments calculated by this hypothesis be small. In integral terms, this condition reduces to a small asymmetry of the random field of surface waves $\eta(\mathbf{x}, t)$:

$$\varepsilon_a = \langle\eta^3\rangle/\langle\eta^2\rangle^{3/2} \ll 1.$$

In view of (3.5), the right-hand side T_1 determined by (3.4) takes the form

$$\begin{aligned} T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}') &= i[2V_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)N(\mathbf{k}_1)N(\mathbf{k}_2) \\ &- V_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}')N(\mathbf{k}_2)N(\mathbf{k}') \\ &- V_2(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}')N(\mathbf{k}_1)N(\mathbf{k}')]\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}'). \end{aligned} \quad (3.6)$$

Thus, we have the explicit expression (3.6) for the right-hand side of equation (3.3) through $N(\mathbf{k})$, and it remains to express the desired third moment $\langle a(\mathbf{k}_1)a(\mathbf{k}_2)a^*(\mathbf{k}_3)\rangle$ through $N(\mathbf{k})$. For this purpose, several equivalent techniques were used in deriving the kinetic equation [2, 8, 15].

The simplest way is the straightforward integration of equation (3.3) with respect to the time with representation (3.6) for T_1 taken into account. Under the zero initial condition, the solution of (3.3) has the form

$$\begin{aligned} &\langle a(\mathbf{k}_1, t)a(\mathbf{k}_2, t)a^*(\mathbf{k}', t)\rangle \\ &= \int_0^t T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}', t-\tau)\exp[i\Delta\sigma(k', -k_1, -k_2)\tau]d\tau. \end{aligned} \quad (3.7)$$

Similarly, the analogues of expressions (3.6) and (3.7) can be found for the other five cumulants appearing on the right-hand side of (3.2). For the third and fifth cumulants, we obtain

$$\begin{aligned} &\langle a^*(\mathbf{k}_1, t)a(\mathbf{k}_2, t)a^*(\mathbf{k}', t)\rangle \\ &= \int_0^t T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}', t-\tau)\exp[i\Delta\sigma(k', +k_1, -k_2)\tau]d\tau, \end{aligned} \quad (3.8)$$

$$\begin{aligned} T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}') &= i[2V_1(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}')N(\mathbf{k}_1)N(\mathbf{k}') \\ &- V_2(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2)N(\mathbf{k}_1)N(\mathbf{k}_2) \\ &- V_2(\mathbf{k}_1, \mathbf{k}', \mathbf{k}_2)N(\mathbf{k}')N(\mathbf{k}_2)]\delta(\mathbf{k}'+\mathbf{k}_1-\mathbf{k}_2), \end{aligned} \quad (3.9)$$

$$\begin{aligned} &\langle a^*(\mathbf{k}_1, t)a^*(\mathbf{k}_2, t)a^*(\mathbf{k}', t)\rangle \\ &= \int_0^t T_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}', t-\tau)\exp[\Delta\sigma(k', k_1, k_2)\tau]d\tau, \end{aligned} \quad (3.10)$$

$$\begin{aligned} T_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}') &= 2V_3(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2)[N(\mathbf{k}_1)N(\mathbf{k}_2) \\ &+ N(\mathbf{k}')N(\mathbf{k}_1) + N(\mathbf{k}')N(\mathbf{k}_2)]\delta(\mathbf{k}'+\mathbf{k}_1+\mathbf{k}_2). \end{aligned} \quad (3.11)$$

The second, fourth, and sixth cumulants on the right-hand side of (3.2) are found from the conjugates of formulas (3.6)–(3.11) with $\mathbf{k}'$ replaced by $\mathbf{k}$.

The substitution of the above expressions for the third moments into the right-hand side of equation (3.1) leads finally to the following evolutionary equation for the spectrum $N(\mathbf{k}, t)$:

$$\frac{\partial N(\mathbf{k}, t)}{\partial t} = 4\mathrm{Re}\int_0^t d\tau\int d\mathbf{k}_1\int d\mathbf{k}_2$$

$$\times \{R_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t')\exp[i\Delta\sigma(k, -k_1, -k_2)\tau] + R_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t')\exp[i\Delta\sigma(k, k_1, -k_2)\tau] + R_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t')\exp[i\Delta\sigma(k, k_1, k_2)\tau]\}. \quad (3.12)$$

Here, $t' = t - \tau$,

$$R_1(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t') = V_1^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \times N(\mathbf{k}, t')N(\mathbf{k}_1, t')N(\mathbf{k}_2, t') \times \left[\frac{1}{N(\mathbf{k}, t')} - \frac{1}{N(\mathbf{k}_1, t')} - \frac{1}{N(\mathbf{k}_2, t')}\right], \quad (3.13)$$

$$R_2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t') = \frac{1}{2}V_2^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) \times N(\mathbf{k}, t')N(\mathbf{k}_1, t')N(\mathbf{k}_2, t') \times \left[\frac{1}{N(\mathbf{k}, t')} + \frac{1}{N(\mathbf{k}_1, t')} - \frac{1}{N(\mathbf{k}_2, t')}\right], \quad (3.14)$$

$$R_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t') = V_3^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) \times N(\mathbf{k}, t')N(\mathbf{k}_1, t')N(\mathbf{k}_2, t') \times \left[\frac{1}{N(\mathbf{k}, t')} + \frac{1}{N(\mathbf{k}_1, t')} + \frac{1}{N(\mathbf{k}_2, t')}\right]. \quad (3.15)$$

4. QUASI-KINETIC APPROXIMATION

To perform integration with respect to the time on the right-hand side of equation (3.2) obtained for $N(\mathbf{k})$, we introduce the effective rate of wave attenuation due to nonlinear interactions $\beta(\mathbf{k})$. The physical meaning of $\beta(\mathbf{k})$ and its formal definition will be discussed below. A specified value of $\beta(\mathbf{k})$ controls the characteristic time of nonlinear interactions $T_\beta(\mathbf{k}) = 2\pi/\beta(\mathbf{k})$ for the $\mathbf{k}$-component of the wave spectrum.

We will use equation (3.12) to describe the evolution of the spectrum $N(\mathbf{k}, t)$ on time scales that are markedly greater than the characteristic time of nonlinear interactions. In this case, we can assume that $t \gg T_\beta(\mathbf{k})$ on the left-hand side of equation (3.12). On the other hand, the integral with respect to the time on the right-hand side of (3.12) must be "saturated" on an interval $\tau < T_\beta(\mathbf{k})$. If $t \gg T_\beta(\mathbf{k})$ and $\tau \le T_\beta(\mathbf{k})$ in the integral of (3.8), one can set

$$R_i(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t - \tau) \cong R_i(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, t), \quad i = 1, 2, 3, \quad (4.1)$$

which makes it possible to factor these functions outside the integral sign. As a result, equation (3.12) takes the following form at times $t \gg T_\beta(\mathbf{k})$:

$$\frac{\partial N(\mathbf{k})}{\partial t} = 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\{V_1^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)N(\mathbf{k})N(\mathbf{k}_1) \times N(\mathbf{k}_2)\left[\frac{1}{N(\mathbf{k})} - \frac{1}{N(\mathbf{k}_1)} - \frac{1}{N(\mathbf{k}_2)}\right]\delta_\beta(\Delta\sigma(k, -k_1, -k_2)) \times \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) + \frac{1}{2}V_2^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2) \times \left[\frac{1}{N(\mathbf{k})} + \frac{1}{N(\mathbf{k}_1)} - \frac{1}{N(\mathbf{k}_2)}\right]\delta_\beta(\Delta\sigma(k, k_1, -k_2)) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) + V_3^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2) \times \left[\frac{1}{N(\mathbf{k})} + \frac{1}{N(\mathbf{k}_1)} + \frac{1}{N(\mathbf{k}_2)}\right] \times \delta_\beta(\Delta\sigma(k, k_1, k_2))\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\}. \quad (4.2)$$

We notice that all the functions $N(\mathbf{k}_i)$ on the right-hand side of equation (4.2) are attributed to the same time t as $N(\mathbf{k})$ standing on its left-hand side. In addition, we have in (4.2) that

$$\delta_\beta(\Delta\sigma(k, \pm k_1, \pm k_2)) = \mathrm{Re}\int_0^{T_\beta} d\tau\exp[i\Delta\sigma(k, \pm k_1, \pm k_2)\tau] = \frac{1}{\pi}\frac{\sin[2\pi\Delta\sigma(k, \pm k_1, \pm k_2)/\beta(\mathbf{k})]}{\Delta\sigma(k, \pm k_1, \pm k_2)} \quad (4.3)$$

represents the spread δ-function, which satisfies to the limit

$$\lim_{\beta \to 0}\delta_\beta(\Delta\sigma(k, \pm k_1, \pm k_2)) = \delta[\Delta\sigma(k, \pm k_1, \pm k_2)]. \quad (4.4)$$

The exact analytic result of integrating with respect to the time (4.3) can be approximated by a more convenient formula. For this purpose, we use the following asymptotic relation, which is valid within the class of generalized functions [15]:

$$\lim_{T_\beta \to \infty}\int_0^{T_\beta}\exp[i(\Delta\sigma\tau)]d\tau = \lim_{\beta \to 0}\frac{i}{\Delta\sigma + i\beta}. \quad (4.5)$$

If β is small but finite, this relation is valid only approximately. However, from this relation, it immediately follows that the limitation of the upper limit of integration with respect to τ is equivalent to adding a small attenuation β to the resonance frequency mismatch $\Delta\sigma$. Therefore, in place of the exact relation (4.3), we can use its approximation by the Lorentz formula

$$\delta_\beta(\Delta\sigma(k, \pm k_1, \pm k_2)) = \frac{1}{\pi}\frac{\beta(\mathbf{k})}{[\Delta\sigma(k, \pm k_1, \pm k_2)]^2 + \beta(\mathbf{k})^2}. \quad (4.6)$$

Formula (4.6) is physically more descriptive in comparison with (4.3) and is more convenient for numerical analysis.

The conventional three-wave kinetic equation is obtained from (4.2) and (4.3) by passing to the limit given in (4.4). In view of $\sigma(k)$ positiveness, the terms with the multiplier V_3^2 on the right-hand side of equation (4.2) drop out, and this equation takes the well-known form [15, 16]

$$\begin{aligned}\frac{\partial N(\mathbf{k})}{\partial t} &= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\{V_1^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\\ &\times N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\left[\frac{1}{N(\mathbf{k})}-\frac{1}{N(\mathbf{k}_1)}-\frac{1}{N(\mathbf{k}_2)}\right]\\ &\times\delta(\sigma(k)-\sigma(k_1)-\sigma(k_2))\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{1}{2}V_2^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\\ &\times\left[\frac{1}{N(\mathbf{k})}+\frac{1}{N(\mathbf{k}_1)}-\frac{1}{N(\mathbf{k}_2)}\right]\\ &\times\delta(\sigma(k)+\sigma(k_1)-\sigma(k_2))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\}.\end{aligned}\tag{4.7}$$

This result is associated with the assumption that the characteristic time of nonlinear interactions $T_\beta = (2\pi/\beta) \longrightarrow \infty$. However, in reality, the quantities T_β and $\beta(\mathbf{k})$ are finite and are determined by equation (4.2). This point is revealed if equation (4.2) is written as

$$\frac{\partial N(\mathbf{k})}{\partial t} = A(\mathbf{k})-\beta(\mathbf{k})N(\mathbf{k}),\tag{4.8}$$

where

$$\begin{aligned}A(\mathbf{k}) &= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2 N(\mathbf{k}_1)N(\mathbf{k}_2)\\ &\times[V_1^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_\beta(\Delta\sigma(k,-k_1,-k_2))\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{1}{2}V_2^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_\beta(\Delta\sigma(k,k_1,-k_2))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\\ &+V_3^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta_\beta(\Delta\sigma(k,k_1,k_2))\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)],\end{aligned}\tag{4.9}$$

and $\beta(\mathbf{k})$ has the form

$$\begin{aligned}\beta(\mathbf{k}) &= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2[V_1^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)[N(\mathbf{k}_1)+N(\mathbf{k}_2)]\\ &\times\delta_\beta(\Delta\sigma(k,-k_1,-k_2))\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{1}{2}V_2^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)[N(\mathbf{k}_1)-N(\mathbf{k}_2)]\\ &\times\delta_\beta(\Delta\sigma(k,k_1,-k_2))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\\ &-V_3^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)[N(\mathbf{k}_1)+N(\mathbf{k}_2)]\\ &\times\delta_\beta(\Delta\sigma(k,k_1,k_2))\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)].\end{aligned}\tag{4.10}$$

The positive term $A(\mathbf{k})$ and the term $\beta(\mathbf{k})N(\mathbf{k})$ on the right-hand side of (4.8) describe the pumping and attenuation of the $\mathbf{k}$-component of a wave spectrum due to nonlinear interactions, respectively. Therefore, the functional $\beta(\mathbf{k})$ in (4.8) has the meaning of the nonlinear wave-attenuation rate, which was formally introduced in deriving equation (4.2). It is seen from the structure of equation (4.10) that $\beta(\mathbf{k}) \sim \varepsilon^2\sigma(\mathbf{k})$, where ε is the characteristic parameter of wave-spectrum nonlinearity.

Thus, equations (4.2), (4.3), and (4.10) form a closed system of equations for a joint determination of the wave spectrum $N(\mathbf{k})$ and the frequency-resonance spreading parameter $\beta(\mathbf{k})$. This system of equations for $N(\mathbf{k})$ and $\beta(\mathbf{k})$ will be referred to as a quasi-kinetic approximation. The quasi-kinetic approximation is more general in comparison with the conventional three-wave kinetic equation (4.7), which is obtained from (4.2) and (4.3) through passing to the limit given in (4.4) and which does not take into account the actual frequency-resonance spreading due to nonlinear interactions.

The above derivation of the quasi-kinetic equation (4.2) for $N(\mathbf{k})$ from the initial dynamic equation (2.4) for $a(\mathbf{k})$ did not use the condition of wave-spectrum narrowness. Under this condition, which implies the smallness of the frequency-resonance mismatch (1.8), the last term on the right-hand side of the quasi-kinetic equation (4.2) can be omitted. Actually, if the wave spectrum $N(\mathbf{k})$ is narrow, the wave vectors $\mathbf{k} = (k_x, k_y)$ are confined within a small vicinity $\Delta k_y(k_x)$ of the main direction of wave propagation $k_x > 0$. For this reason, the resonance condition $\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2 = 0$ cannot be satisfied for a narrow wave spectrum, and the last term on the right-hand side of (4.2) vanishes. As a result, the quasi-kinetic equation (4.2) takes the form

$$\begin{aligned}\frac{\partial N(\mathbf{k})}{\partial t} &= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\{V_1^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\\ &\times N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\left[\frac{1}{N(\mathbf{k})}-\frac{1}{N(\mathbf{k}_1)}-\frac{1}{N(\mathbf{k}_2)}\right]\\ &\times\delta_\beta(\Delta\sigma(k,-k_1,-k_2))\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{1}{2}V_2^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\\ &\times\left[\frac{1}{N(\mathbf{k})}+\frac{1}{N(\mathbf{k}_1)}-\frac{1}{N(\mathbf{k}_2)}\right]\\ &\times\delta_\beta(\Delta\sigma(k,k_1,-k_2))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\}.\end{aligned}\tag{4.2'}$$

We notice that the smallness of the frequency-resonance mismatch (1.8) is violated for the omitted term. The corresponding expression for $\beta(\mathbf{k})$ (4.10) will take the form

$$\begin{aligned}\beta(\mathbf{k}) &= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\{V_1^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)[N(\mathbf{k}_1)+N(\mathbf{k}_2)]\\ &\times\delta_\beta(\Delta\sigma(k,-k_1,-k_2))\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\\ &+\frac{1}{2}V_2^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)[N(\mathbf{k}_1)-N(\mathbf{k}_2)]\\ &\times\delta_\beta(\Delta\sigma(k,k_1,-k_2))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\}.\end{aligned}\tag{4.10'}$$

The quasi-kinetic approximation also arises as an intermediate stage in deriving the four-wave kinetic equation (1.1). Following the above procedure, one can verify that (1.1) is replaced in this approximation by

$$\frac{\partial N(\mathbf{k})}{\partial t} = P_\beta(\mathbf{k})$$
$$= 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\int d\mathbf{k}_3 T^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$$
$$\times\{N(\mathbf{k}_2)N(\mathbf{k}_3)[N(\mathbf{k})+N(\mathbf{k}_1)] - N(\mathbf{k})N(\mathbf{k}_1)[N(\mathbf{k}_2)+N(\mathbf{k}_3)]\} \quad (4.11)$$
$$\times\delta_\beta(\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3),$$

where

$$\delta_\beta(\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3)) \quad (4.12)$$
$$= \frac{1}{\pi}\frac{\sin[2\pi(\Delta(\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3))/\beta(\mathbf{k})]}{\delta_\beta(\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3)},$$

$$\beta(\mathbf{k}) = 4\pi\int d\mathbf{k}_1\int d\mathbf{k}_2\int d\mathbf{k}_3 T^2(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$$
$$\times[N(\mathbf{k}_3)N(\mathbf{k}_1)+N(\mathbf{k}_1)N(\mathbf{k}_2)-N(\mathbf{k}_2)N(\mathbf{k}_3)] \quad (4.13)$$
$$\times\delta_\beta(\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3))\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3).$$

The conventional four-wave kinetic equation (1.1) is obtained from these relations through passing to the limit $\beta \longrightarrow 0$ and ignoring the actual definition of $\beta(\mathbf{k})$ by relation (4.13).

5. DISCUSSION

In a four-wave case, when the resonance conditions

$$\sigma(k)+\sigma(k_1)-\sigma(k_2)-\sigma(k_3) = 0,$$
$$\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3 = 0$$

are necessarily satisfied, the actual distinctions between the quasi-kinetic system of equations (4.11)–(4.13) and the conventional kinetic equation (1.1) are insignificant; a weak spreading of δ-functions in equation (4.11) is on the order of $\beta(\mathbf{k}) \sim \varepsilon^4\sigma$ and leads to a small difference between the integrals appearing in (4.11) and (1.1).

However, calculating the nonlinear attenuation rate $\beta(\mathbf{k})$ in a four-wave case is important, for example, in estimations of the nonlinear spreading of the space–time wave spectrum $E(\mathbf{k}, \omega)$ about the dispersion surface $\sigma(k)$ (see, for example, [17]). For this purpose, as a first approximation, one can use a simplified formula for $\beta(\mathbf{k})$ that is obtained by passing to the limit $\beta \longrightarrow 0$ on the right-hand side of relation (4.13).

A similar situation occurs when the three-wave quasi-kinetic approximation is used to describe waves with a dispersion function $\sigma(k)$ that obeys the three-wave resonance conditions (1.4)—for example, capillary surface waves with $\sigma(k) \sim k^{3/2}$. In this case, the solution of the quasi-kinetic system of equations (4.2), (4.3), and (4.10) must also be close to that of the conventional three-wave equation (4.7).

A different situation arises when the dispersion function $\sigma(k)$ does not satisfy the three-wave resonance conditions (1.4), but the frequency-resonance mismatch turns out to be small for a number of reasons. We notice that this is the case for a weak dispersion described by the Boussinesq approximation (1.6). In this case, from the conventional three-wave kinetic equation, it follows that $\partial N(\mathbf{k})/\partial t \equiv 0$. However, the corresponding quasi-kinetic system of equations determines the evolution of the wave spectrum $N(\mathbf{k})$ that is due to nonresonance three-wave interactions.

It should be noted that the concept of introducing quasi-resonance interactions into the evolutionary equation for a wave spectrum has already been discussed in the literature [18]. In place of relation (3.5), study [18] proposed a specific splitting of fourth moments such that an imaginary addition to the frequency appeared in equations for third moments. Finally, equations for wave spectra took an almost identical form as equation (4.2'), although the derivation of those equations was somewhat different from that used in our work.

Although the imaginary part of the frequency $\beta(\mathbf{k})$ can be formally defined in different ways, the main concept of taking into account a finite interaction time in the kinetic integral itself allows a correct description of three-wave processes in a finite-depth water.

In conclusion, we note the importance of oceanological aspects of the problem under consideration. Shallow-water areas and coastal regions are the places of oil platforms arranged, as a rule, at depths of a few tens of meters. Their design calls for calculations of wind waves in storms extreme for a given region. Assuming that wind velocities in such storms are estimated at $U_a \simeq 30$ m/s and that fetches are large enough for the regime of developed waves with a dimensionless wave number of a spectral maximum $\tilde{k}_m = k_m U^2 g \simeq 1$ to be reached, we find that $k_m \simeq 0.01$ rad/m. In this case, for example, at depths h ranging from 10 to 50 m, the dimensionless combination $k_m h$ varies from 0.1 to 0.5. Therefore, even those water areas that are considered to be deep-water areas under mean climatic winds turn out to be shallow-water areas in such extreme situations. For example, at $U_a \simeq 10$ m/s and $\tilde{k}_m \simeq 1$, we obtain the estimate $k_m h = 1$–5, which corresponds to a deep-water case.

At $k_m h = 0.1$–0.5, conventional methods of wind-wave prediction, which use a four-wave integral of nonlinear interactions, are no longer applicable. A search for other mechanisms of nonlinear interactions of gravity waves at small depths has stimulated this work, which is the first step in the development of a model for wind-wave evolution at small depths.

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