Nonlinear Processes in Geophysics (2002) 9: 355–366

Nonlinear Processes in Geophysics

A basing of the diffusion approximation derivation for the four-wave kinetic integral and properties of the approximation

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Received: 5 March 2001 – Revised: 23 July 2001 – Accepted: 22 January 2002

Abstract. A basing of the diffusion approximation derivation for the Hasselmann kinetic integral describing nonlinear interactions of gravity waves in deep water is discussed. It is shown that the diffusion approximation containing the second derivatives of a wave spectrum in a frequency and angle (or in wave vector components) is resulting from a step-by-step analytical integration of the sixfold Hasselmann integral without involving the quasi-locality hypothesis for nonlinear interactions among waves. A singularity analysis of the integrand expression gives evidence that the approximation mentioned above is the small scattering angle approximation, in fact, as it was shown for the first time by Hasselmann and Hasselmann (1981). But, in difference to their result, here it is shown that in the course of diffusion approximation derivation one may obtain the final result as a combination of terms with the first, second, and so on derivatives. Thus, the final kind of approximation can be limited by terms with the second derivatives only, as it was proposed in Zakharov and Pushkarev (1999). For this version of diffusion approximation, a numerical testing of the approximation properties was carried out. The testing results give a basis to use this approximation in a wave modelling practice.

1 Introduction

A kinetic integral for the spectrum evolution rate due to nonlinear interaction among waves derived by Hasselmann (1962) has the kind

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} \equiv T_N(k) = 4\pi \int d\boldsymbol{k}_1 \int d\boldsymbol{k}_2 \int d\boldsymbol{k}_3 \cdot M^2(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) N(\boldsymbol{k}) N(\boldsymbol{k}_1) N(\boldsymbol{k}_2) N(\boldsymbol{k}_3) \cdot \left[\frac{1}{N(\boldsymbol{k})} + \frac{1}{N(\boldsymbol{k}_1)} - \frac{1}{N(\boldsymbol{k}_2)} - \frac{1}{N(\boldsymbol{k}_3)} \right] \cdot \delta(\sigma(k) + \sigma(k_1) - \sigma(k_2) - \sigma(k_3)) \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) . \quad (1)$$

Here, $T_N(\boldsymbol{k})$ is the nonlinear transfer for the wave action spectrum, $N(\boldsymbol{k})$, $\boldsymbol{k}$ is the wave vector and $\sigma(k)$ is the frequency of the same wave component, given by the dispersion law for deep water of the kind $\sigma_i^2 = gk_i$; $M(...)$ is the four-wave interaction matrix element, and $\delta(...)$ is the Dirac delta-function of its argument. It is evident that use of the sixfold integral (1) in its explicit form for the problem of wave evolution is very awkward due to the difficulty of its direct calculation. For this reason, it is rather natural to seek an effective approximation for the integral (1), which retains the main properties of the latter.

Despite a number of attempts to construct approximations for the Hasselmann kinetic integral (see, for example, Hasselmann et al., 1985; Efimov and Polnikov 1991), the regular analytical derivations of approximations are innumerable. Thus, the problem of analytical derivation of an effective approximation for the integral (1) is still important. A recent paper by Zakharov and Pushkarev (1999) can serve as proof of this statement. This circumstance is provided by the principal role of the kinetic integral for a description of wind wave evolution. From this point of view, a finding of an analytical approximation derived by rigorous algebra transformations of Eq. (1) without fitting parameters has a special interest.

At present, in fact, there are only two variants of analytical approximations derived from the exact formula for the kinetic integral (1) on the basis of direct algebra transformations. One of them is derived by

Hasselmann and Hasselmann (1981)[1] and is called the diffusion approximation. The second approximation is derived by Zakharov and Smilga (1981). Later it was conventionally called "the narrow directional approximation" (Zaslavskii, 1989). Here we shall not dwell on the second approximation, noting only that it is applicable for wave spectra with narrow angular spreading functions.

A domain of applicability for the diffusion approximation is more wider. But use of it in the form proposed in paper I

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[1]Hereafter is referred to as paper I.

meets a series of technical difficulties connected to the calculation of the fourth derivatives for the spectral function $N(\boldsymbol{k})$. In this aspect, it seems very interesting to simplify the final expression for the diffusion approximation to a level of the second derivatives for $N(\boldsymbol{k})$, as it was proposed in Zakharov and Pushkarev (1999)[2]. Herewith, we should note that their approximation is based on intuitive considerations using the conservation properties of the exact integral (1) but not on algebra transformations of Eq. (1). Therefore, the necessity appears to find the result of paper II by means of direct algebra transformations of the integral (1).

In present paper, we make a basing of the diffusion approximation (in terms of a combination of the zero, first, and second derivatives of the spectrum $N(\boldsymbol{k})$) by means of consecutive analytical integration of the integral (1). In this way, we are lucky to find a convenient kind of approximation in the form close to the one proposed in paper II and to give a clear physical treatment to the final result as the small scattering angle approximation. We remind the reader that just such a type of diffusion approximation treatment was initially given in paper I. But, in difference to paper I, here it will be shown that there is no necessity to introduce the hypothesis of locality for nonlinear interactions among waves, as far as the type of integrand singularities provides itself the small scattering angle approximation. Thus, we are lucky to unite the main ideas of papers I and II.

The structure of the paper is the following. In Sect. 2, following paper I, a short analysis of the diffusion approximation derivation is given. In Sect. 3, the main ideas of the same approximation are discussed, following paper II. An alternative direct derivation of the diffusion approximation is presented in Sect. 4. Section 5 is devoted to a numerical testing of the diffusion approximation given in terms of paper II. For this aim, we compare the approximate nonlinear transfer with the numerical results of direct calculations for the integral (1) obtained in Polnikov (1989, 1990). In Sect. 6, the final analysis is carried out and conclusions are drawn.

2 Analysis of the diffusion approximation derivation by Hasselmann and Hasselmann

The main ideas of the diffusion approximation derivation given in paper I are as follows.

1. On the basis of direct calculations of the nonlinear energy transfer, $T_E(\boldsymbol{k}) \propto \sigma^{-1} T_N(\boldsymbol{k})$, it is proposed that the main contribution to the transfer is made by the wave components with wave vectors $\boldsymbol{k}$ located in the domain $\boldsymbol{k}_1 \approx \boldsymbol{k}_2 \approx \boldsymbol{k}_3 \approx \boldsymbol{k}$ (the locality hypothesis for nonlinear interactions).

2. The spectra $N(\boldsymbol{k})$ are expanded in the small vectors, $\boldsymbol{k}' = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and $\boldsymbol{k}'' = \boldsymbol{k}_3 - \boldsymbol{k}$. After this, the cubic spectral function under the integral is transformed to an expression containing the second derivatives of spectral function in the wave vectors components, and the final integral obtains the form of an integral with respect to the small values $\boldsymbol{k}'$ and $\boldsymbol{k}''$, which has a rather strange mathematical sense.

3. The delta-function for wave vectors is expanded in the same small vectors. As a result, the final expression for the transfer contains the fourth derivatives of the kind

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} = \frac{\partial^2}{\partial k_i \partial k_j} \left\{ D_{ijlm} \left[N^2 \frac{\partial^2 N}{\partial k_i \partial k_j} - 2N \frac{\partial N}{\partial k_l} \frac{\partial N}{\partial k_m} \right] \right\}, \quad (2)$$

where k_i, k_j are the wave vector components, and

$$D_{ijlm} = 2^{-9} \int \int (k'_l k'_m - k''_l k''_m)\, k''_i k''_j \, R \, d\boldsymbol{k'} \, d\boldsymbol{k''}. \quad (3)$$

Herewith, the factor R in the integrand (3) contains the frequency delta-function.

4. Using of symmetry properties of the integrand and the conservation laws for the total wave action

$$A = \int N(\boldsymbol{k})\, d\boldsymbol{k}, \quad (4)$$

energy

$$E = \int \sigma(k)\, N(\boldsymbol{k})\, d\boldsymbol{k}, \quad (5)$$

and momentum

$$\boldsymbol{M} = \int \boldsymbol{k}\, N(\boldsymbol{k})\, d\boldsymbol{k} \quad (6)$$

results in an essential simplification of Eq. (2), leading to a decrease in the independent elements number for the tensor D_{ijlm}.

5. For the case of deep water, the situation is simplified additionally, as far as there is no need to calculate the integrals (3) directly. Instead, one can use a simple relation of the kind $D_{ijlm} \propto C_{ijlm} k^b$ with the value b determined from the dimension consideration.

Finally, the expression for the transfer obtains the kind

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} = C_1 \left\{ \left(\nabla^2 + \frac{\partial^2}{\partial k_i \partial k_j} \frac{k_i k_j}{K^2} \right) \right\} (F_1 + F_2) + C_2 \left\{ \left(2 \frac{\partial^2}{\partial k_i \partial k_j} \frac{k_i k_j}{k^2} \right) F_1 + \frac{\partial^2}{\partial k_i \partial k_j} F_3 + \left(2\nabla^2 - \frac{\partial^2}{\partial k_i \partial k_j} \frac{k_i k_j}{K^2} \right) F_2 \right\}, \quad (7)$$

[2] Hereafter is referred to as paper II.

where

$$F_1 = g^{\frac{3}{2}} k^{\frac{27}{2}} \left\{ N^2 \nabla^2 N - 2N(\nabla N)^2 \right\}, \quad (8)$$

$$F_2 = g^{\frac{3}{2}} k^{\frac{27}{2}} \left\{ \frac{k_l k_m}{k^2} \left(N^2 \frac{\partial^2 N}{\partial k_l \partial k_m} - 2N \frac{\partial N}{\partial k_l} \frac{\partial N}{\partial k_m} \right) \right\}, \quad (9)$$

$$F_3 = g^{\frac{3}{2}} k^{\frac{27}{2}} \left(N^2 \frac{\partial^2 N}{\partial k_l \partial k_m} - 2N \frac{\partial N}{\partial k_l} \frac{\partial N}{\partial k_m} \right). \quad (10)$$

The coefficients C_1 and C_2 are determined from a comparison with exact calculations.

Thus, the derivation in paper I includes the following assumptions: one hypothesis (item 2.1), a rather doubtable integral with respect to small variables of the type (3) (item 2.2), and the expansion of the wave vector delta-function, which is not clear in the domain of applicability (item 2.3). All these items decrease the reliability of the final result. Herewith, for justification, one should note that the ideas 2.4. and 2.5. are very fruitful.

Finally, Eq. (7) is the analytical approximation for the integral (1), derived by the rather rigorous, though too sophisticated algebra transformations. Its testing carried out in paper I showed that this approximation corresponds qualitatively to the exact one-dimensional energy transfers obtained in paper I by numerical calculations of the integral (1). The approximation (7) reproduces all principal features of the exact transfer with the mean error of the order of 30–50% (depending on the spectrum shape).

The main drawback of the parameterization (7) is a complication of the final formula. In this aspect, the approximation of paper II is more preferable, if it is correctly derived. To check this point, we try to analyze the derivation in paper II.

3 Analysis of the diffusion approximation derivation by Zakharov and Pushkarev

The main ideas of the diffusion approximation derivation given in paper II differ absolutely from the ones given in paper I. They are as follows:

1. First, the fact of conservation of the total wave action, A, wave energy, E, and wave momentum, $\boldsymbol{M}$, is stated as the main feature of the integral (1).

2. Second, there is noted the fact of the exact stationary solutions' existence for the Eq. (1), including the so-called thermodynamic solution of the kind

$$N(\boldsymbol{k}) = \frac{a}{(b + \sigma(k))}, \quad (11)$$

(where a and b are arbitrary constants) and the Kolmogorov-type solutions of the kind

$$N(\boldsymbol{k}) \propto k^{-23/6}, \quad (12)$$

$$N(\boldsymbol{k}) \propto k^{-4}, \quad (13)$$

corresponding to a constant flux of wave action to lower frequencies (the spectrum (12)) and a constant flux of wave energy to higher frequencies (the spectrum (13)).

3. Then, in the polar coordinates, (σ, θ), the simplest differential operator of the kind

$$L = \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} + \frac{1}{\sigma^2} \frac{\partial^2}{\partial \theta^2} \quad (14)$$

is constructed, which secures the conservation laws' validity for values A, E, and $\boldsymbol{M}$. Additionally, this operator retains the solution (11), if the initial Eq. (1) is imitated by a model equation of the kind

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} = \frac{1}{\sigma^3} L \left[\frac{1}{N(\boldsymbol{k})} \right]. \quad (15)$$

4. In a more general form (taking into account the cubic power of the integrand in the spectral function $N(\boldsymbol{k})$, the dimension of the matrix elements, and retention of the Kolmogorov-type solutions), the imitative representation of the kinetic equation obtains the following type of differential equation

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} = \frac{c}{\sigma^3} L \left[N^4(\boldsymbol{k}) \sigma^{26} L \left[\frac{1}{N(\boldsymbol{k})} \right] \right]. \quad (16)$$

5. In final form, Eq. (16) is simplified to a second order differential equation of the kind

$$\frac{\partial N(\boldsymbol{k})}{\partial \boldsymbol{t}} = \frac{c'}{\sigma^3} L \left[N^3(\boldsymbol{k}) \sigma^{24} \right], \quad (17)$$

if one supposes that the spectrum shape is of the kind $N(\boldsymbol{k}) \propto \sigma^{-b} \Phi(\sigma, \theta)$, where the function $\Phi(\sigma, \theta)$ is weakly dependent on frequency. Note that in Eqs. (16) and (17), the dimensional constants, c and c', depend on the gravity acceleration, g.

Thus, in paper II, there are in fact no analytical transformations of the initial integral (1), but the conservation properties of Eq. (1) are only used. In this respect, the method of construction of the kinetic integral parameterization proposed in paper II is rather artificial. It is sooner the discovered imitation of the kinetic integral, in the framework of which a real structure of the integral is not used. The state of the authors of paper II, that Eqs. (16) and (7) are equivalent, is not obvious directly. It seems that this state needs special testing.

Due to the circumstance stated above, it is worthwhile to ground Eq. (17) by means of direct algebra transformations of the integral (1). In the case when the result (17) is justified, it needs to test a quality of this approximation on the basis of direct comparison of a two-dimensional transfer calculated by the approximation (17), with an exact calculation of the integral (1). These two problems are solved below in Sects. 4 and 5 of this paper.

4 Alternative derivation of the diffusion approximation

The main idea of the approach proposed here is based on an analytical part of the numerical calculation algorithm for the integral (1), derived in Polnikov (1989). In accordance with the algorithm mentioned, the analytical transformations are carried out by the following steps.

1. First, both delta-functions under the integral are integrated absolutely exact. For this aim, one starts from the integration in the vector $\boldsymbol{k}_2$, which eliminates the wave vector delta-function. Then, a transition to the polar coordinates, (σ, θ), is carried out, and the exact analytical integration in the angle θ_1 for the frequency delta-function is executed. As a result, the sixfold integral becomes the threefold one of the kind

$$\frac{\partial N(\sigma,\theta)}{\partial t} = c \iiint M^2(...) \sum\nolimits_{\pm} P_3 \left[N(\sigma,\theta), N(\sigma_1,\theta_1), N(\sigma_2,\theta_2^{\pm}), N(\sigma_1,\theta_1)\right] \frac{J}{k_2}\, d\sigma_1\, d\sigma_3\, d\theta_3\,, \tag{18}$$

where c is the dimensional coefficient depending on g, $P_3(\dots)$ is a cubic form of the kind

$$P_3(N, N_1, N_2, N_3) = N_1 N_2 (N + N_3) - N N_3 (N_1 + N_2)\,, \tag{19}$$

(where, in turn, the following notion is used: $N_i = N(\sigma_i, \theta_i)$, and the zero sub-index is omitted), J is the Jacobian of the frequency delta-function integration, k_2 is the modified wave number corresponding to the point $(\sigma_2, \theta_2^{\pm})$ (see details in Polnikov, 1989).

2. An explicit form of J has the kind

$$J = c' \hat{\sigma}_2^3 \left(\left(k_a + k_1 + \hat{\sigma}_2^2\right)^{1/2} \left(k_a - k_1 + \hat{\sigma}_2^2\right)^{1/2} \left[\left(k_a + k_1 - \hat{\sigma}_2^2\right)^{1/2} \left(k_1 + \hat{\sigma}_2^2 - k_a\right)^{1/2} \right] \right)^{-1}. \tag{20}$$

where c' is the dimensional constant depending on g, and

$$\hat{\sigma}_2 = \frac{\sigma_a - \sigma_2}{\sqrt{g}}, \tag{21}$$

$$\sigma_a = \sigma_3 + \sigma = \sigma_1 + \sigma_2, \tag{22}$$

$$k_a = \left[k_3^2 + k^2 + 2kk_3 \cos(\Delta_{30})\right]^{1/2}, \tag{23}$$

$$\Delta_{30} \equiv \theta_3 - \theta. \tag{24}$$

Note that due to the symmetry properties of the integrand, integration in Eq. (18) can be restricted by the domain where $\sigma_1 \le \sigma_2$ (i.e. for $\sigma_1 \le \sigma_a/2$), while the value of the constant c is doubled. For the same reason, one can restrict the integral calculation by the condition $\sigma_3 \ge \sigma$ with the same enhancing of the coefficient c.

3. The Jacobian denominator analysis shows that, under the condition $\sigma_3 \ge \sigma$, integrable singularities arise only due to the factors in the square brackets of the denominator for Eq. (20). By finding the roots of proper equations, one can determine the lower, $\sigma_1^{(1)}$, and the upper, $\sigma_1^{(2)}$, limits for the integral in the variable σ_1. Thus, taking into account that said, Eq. (18) can be represented in the kind

$$\frac{\partial N(\sigma,\theta)}{\partial t} = C \int_0^{2\pi} d\theta_3 \int_{\sigma}^{\infty} d\sigma_3 \int_{\sigma_1^{(1)}}^{\sigma_1^{(2)}} d\sigma_1\, \frac{M^2\, P_3}{\sigma_2 k_2} \left(\left(\sigma_1 - \sigma_1^{(1)}\right) \left(\sigma + o(\Delta_{30}) - \sigma_1\right) \left(\sigma_3 + o(\Delta_{30}) - \sigma_1\right) \right)^{-1/2}, \tag{25}$$

where the lower limit is the following

$$\sigma_1^{(1)} = \frac{\sigma_3 \sigma + o(\Delta_{30})}{\sigma_3 + \sigma} \tag{26}$$

and the upper limit, $\sigma_1^{(2)}$, is determined by the second factor in the denominator of Eq. (25). The term $o(\Delta_{30})$ means that a certain function of the angular difference, $\Delta_{30} \equiv \theta_3 - \theta$, has a small value for small values of Δ_{30}. Additionally, we should note that, for the values $\sigma_3 \approx \sigma$, the following ratio takes place: $o(\Delta_{30}) \approx 0.25(\Delta_{30})^2$. Thus, the "small scattering angle" condition, $o(\Delta_{30}) \ll 1$, is valid up to the values $(\Delta_{30}) \approx 1$, i.e. in a rather large interval of the scattering angles.

4. From the structure of Eq. (25) we can draw the following.

   4.1 With a certain accuracy one can assume that a representative part of the integral is accumulated from the regions of the singular surfaces, i.e. from the regions close to the upper and lower limits in σ_1 of the integral (25)[3]. (Exact calculations showed (Polnikov, 1995) that the contribution of the singular surfaces is about 25–30% of the total integral and has a proper shape.)

   4.2 The main part of the singularity surfaces' contribution to the integral is given by the singularity at the upper limit, $\sigma_1^{(2)}$, of the integral in σ_1, as far as the singularity has a two-dimensional feature and corresponds to small values of Δ_{30} and to the condition $\sigma_1 \approx \sigma \approx \sigma_3$.

   4.3 The latter feature corresponds automatically to the criterion of locality for the nonlinear interactions and to the approximation of small scattering angles.

[3] This idea is also used for the stating of approximations in Zakharov, Smilga (1981) and Hasselmann et al. (1985).

4.4 From that said above, one can draw the main conclusion that the searched integral can be approximated by an analytical estimation of the contribution at the upper limit of the integral (25) in σ_1 (i.e. from the region given by the conditions presented in the item 4.4.2.). This estimation is just the small scattering angle estimation. And the locality interaction condition is merely the consequence of the singularity feature for the integrand, arising without involving any hypothesis.

Furthermore, to finish the approximation derivation, one needs to find a regular part of the integrand for Eq. (25) in the vicinity of the most strong singularity ($\sigma_1 \approx \sigma \approx \sigma_3 \approx \sigma_2$, $\Delta_{30} \approx 0$), which can be denoted by the index $\sum T$, and to execute the rest of the integration. But the cubic form $P_3(...)$ at this surface is equal to zero. Therefore, $P_3(...)$ should be expanded in a series in frequency and angle at the singularity surface, which leads to arising of the terms of the type in the right-hand side for Eq. (2).

A scheme of analytical transformations is as follows. Retaining the cubic form under the integral as a separate factor, we have

$$\frac{\partial N(\sigma,\theta)}{\partial t} = \frac{CM^2\big|_{\Sigma T}}{\sigma^3} \int\limits_0^{2\pi} d\theta_3 \int\limits_\sigma^\infty d\sigma_3 \int\limits_{\sigma_1^{(1)}}^{\sigma_1^{(2)}} d\sigma_1 P_3\big|_{\Sigma T} \cdot (\sigma + o(\Delta_{30}) - \sigma_1)^{-1/2} (\sigma_3 + o(\Delta_{30}) - \sigma_1)^{-1/2} . \quad (27)$$

The Tailor expansion of each spectrum function in the cubic form P_3 at the surface ΣT is of the kind

$$N(\sigma_i,\theta_i) = N^{(0)}(\sigma,\theta) + N_\sigma^{(1)}(\sigma_i - \sigma) + 0.5N_\sigma^{(2)} (\sigma_i - \sigma)^2 + N_\theta^{(1)}(\theta_i - \theta) + 0.5N_\theta^{(2)}(\theta_i - \theta)^2 \dots , \quad (28)$$

where the mixed derivatives are omitted for simplicity. Substitution of Eq. (28) into Eq. (19), accounting for the resonant conditions due to δ-functions under the integral, results in a cancellation of the zero derivatives' contribution. The left terms with derivatives at the surface ΣT can be extracted from the integral, which leads to the expression of the kind

$$\frac{\partial N(\sigma,\theta)}{\partial t} = \frac{CM^2\big|_{\Sigma T}}{\sigma^3} \Bigg[\sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=3}} b_{\alpha\beta\gamma} \sigma^{3-\beta-\gamma} \left(N_\sigma^{(0)}\right)^\alpha \left(N_\sigma^{(1)}\right)^\beta \left(N_\sigma^{(2)}\right)^\gamma + \sum_{\substack{\alpha_1,\beta_1,\gamma_1 \\ \alpha_1+\beta_1+\gamma_1=3}} b_{\alpha_1\beta_1\gamma_1} \sigma^{3-\beta_1-\gamma_1} \left(N_\theta^{(0)}\right)^{\alpha_1} \left(N_\theta^{(1)}\right)^{\beta_1} \left(N_\theta^{(2)}\right)^{\gamma_1} + \dots \Bigg] \quad (29)$$

where coefficients $b_{\alpha\beta\gamma}$ are the integrals of the following kind

$$\int\limits_0^{2\pi} d\theta_3 \int\limits_\sigma^\infty d\sigma_3 \int\limits_{\sigma_1^{(1)}}^{\sigma_1^{(2)}} \frac{(\sigma - \sigma_1)^\beta (\sigma_3 - \sigma_1)^\gamma d\sigma_1}{(\sigma + o(\Delta_{30}) - \sigma_1)^{1/2} (\sigma_3 + o(\Delta_{30}) - \sigma_1)^{1/2}} , \quad (30)$$

analogous to Eq. (30) with a proper change of variables in the numerator under the integral.

As seen from Eq. (29), the final expression is a sum of terms of the third order in spectral function with different orders of derivatives and with proper coefficients. Thus, the form of the approximation is more general than the one for the diffusion approximation presented in papers I and II. Here we should note that these kind of derivatives depend on the kind of variables. They may be as follows: frequency-angle, wave number modulus–angle, or wave vector components. At some conditions (for example, if we use the frequency-angle variables), a contribution of the first derivative terms in Eq. (29) is also cancelled, and the form of the approximation becomes typical to the diffusion one.

The last point of our derivation is to represent the final expression in a compact form, taking into account the conservation properties of the initial integral. To do this, it is very convenient to use the ideas 2.4 and 2.5 proposed in paper I. In our case, by analogy, there is no need to carry out the integration in Eq. (30), as far as the dimension considerations are sufficient. In this way, the coefficients $b_{\alpha\beta\gamma}$ in Eq. (29) may be determined from the conservation laws' conditions for the total values A, E, and $\boldsymbol{M}$.

Finally, one may expect to obtain an expression similar to Eq. (17), though not in the simplest form proposed in paper II. Particularly, the integration in θ_3, taking place in Eq. (25), may lead to a change in the angular dependence of the approximation. But this question deals with a specification of the diffusion approximation and is not fundamental at present. The main fact is that, in principal, we have the rigorous derivation of the diffusion approximation for the integral (1).

A specification of the diffusion approximation can be done in the course of consecutive testing more and more complicated variants of the approximations. At present, it is worth starting the testing from the variant (17). By determining its advantages and drawbacks, one can move to the direction of a more complicated and more effective analytical approximation in the framework of the result (25), if it is justified in practice.

5 The diffusion approximation testing

An order of the approximation (17) testing will be as follows. First, Eq. (17) is transformed to a more convenient type in the energy spectrum $S(\sigma,\theta)$ representation, usually used in

Table 1. Spectrum shapes and calculated nonlinear extremes used for the coefficient c' estimation

Spectrum shape	Positive extreme, T^+		Negative extreme, T^-		c'
	Exact	Approx.	Exact	Approx.	
$S_{PM}(\sigma)\cos^2\theta$	27.8	114	−62.2	−268	0.23±0.10
$S_{PM}(\sigma)\cos^8\theta$	12.4	105	−40.	−429	0.10±0.02
$S_J(\sigma)\cos^2\theta$	4.45	57	−6.35	−142	0.06±0.03
$S_J(\sigma)\cos^8\theta$	1.6	51	−4.8	−181	0.03±0.01

Note: Estimations for T^+ and T^- are given in units of the coefficient (31). They are found by calculations of the integral (1) (exact) and Eq. (35) (approx.). Estimations of c' are given by the ratio (exact)/(approx.).

practice. Herewith, the dimensional coefficient c' is changed by a proper, nondimensional one.

Second, a dimensional factor of the kind

$$c = \frac{\pi}{16} g^{-4} S_p^3 \sigma_p^{11} \tag{31}$$

is extracted from the final expression for $\partial S/\partial t$, as it was introduced earlier in Polnikov (1989), while tabulating the numerical results for the integral (1) (here, σ_p and S_p are the peak frequency and the peak value of the spectrum, respectively). Thus, by a comparison of approximated and "exact" numerical estimations for the transfer $\partial S/\partial t$ (the latter is presented in Efimov and Polnikov, 1991; Polnikov, 1989), one can easily find the nondimensional coefficient in a transformed analytical formula for the diffusion approximation.

Then, a comparison of the two-dimensional transfer for both estimations is carried out, and a quantitative estimation of the approximation error is found. At the final stage, we shall compare a numerical solution of the kinetic equation in the diffusion approximation with the numerical solution of Eq. (1) found in Polnikov (1990).

5.1 Transformation of Eq. (17)

Note that the final formula for the diffusion approximation (17) is very inconvenient for practical use. First, it is not correct in dimension, as the authors proposed in paper II that $g = 1$. Second, in Eq. (17), they used the derivatives in frequency and angle, but the wave action spectrum is presented in the $\boldsymbol{k}$-space. And finally, the wave action spectrum itself does not have its own shape parameterization. This leads to the necessity to use a transformation to the spectrum $S(\sigma,\theta)$, whose shape is well-known and widely used. Just for these reasons, Eq. (17) should be rewritten in the $S(\sigma,\theta)$ representation, at least for the testing problems.

Using the definition

$$N(\boldsymbol{k})d\boldsymbol{k} = \frac{\gamma\, g}{\sigma}\, S(\sigma,\theta)\, d\sigma\, d\theta\,, \tag{32}$$

where $\gamma = 4\pi^2$, it is easy to show that

$$N(\boldsymbol{k}) = \frac{\gamma\, g^3}{2\sigma^4}\, S(\sigma,\theta). \tag{33}$$

Substitution of Eq. (33) into Eq. (17) gives

$$\frac{\partial S}{\partial t} \equiv T(\sigma,\theta) = \tilde{c} g^{-4} \sigma\; L\left[\sigma^{12} S^3(\sigma,\theta)\right], \tag{34}$$

where a proper power of g and a new, nondimensional coefficient $\tilde{c}$ is introduced. Note that the nonlinear energy transfer itself is denoted as $T(\sigma,\theta)$.

5.2 Nondimensional coefficient estimation

To determine the nondimensional coefficient for the diffusion approximation, let us rewrite Eq. (34) in the kind

$$T(\sigma,\theta) = \frac{\pi}{16} g^{-4} S_p^3 \sigma_p^{11} c' \left\{ \frac{16}{\pi} \sigma_p^2 \hat{\sigma}\; L\left[\hat{\sigma}^{12} \hat{S}^3(\sigma,\theta)\right]\right\}, \tag{35}$$

where c' is the sought after coefficient, and the expression in the figure brackets is the modified diffusion operator in which "the hat" means that frequencies and spectra are normalized by their values at the peak point.

As far as the nonlinear transfer, $T(\sigma,\theta)$ is a two-dimensional function, where a certain procedure of minimization should be used to determine a value of the constant c'. For simplicity, we use the mean value method for estimations of c' obtained by the ratio

$$c' = T^{\pm}_{exa} / T^{\pm}_{apr}$$

for two extreme values of the two-dimensional transfer $T(\sigma,\theta)$: the positive extreme, T^+, and the negative one, T^-. Herewith, $T^{\pm}_{apr}$ is calculated by Eq. (35) without coefficient c', and is the value tabulated in Efimov and Polnikov (1991) for the cases of direct calculations for the integral (1).

In present calculations the standard JONSWAP shape of the spectrum is used:

$$S(\sigma,\theta) = C\sigma^{-5} \exp\left[-1.25\left(\frac{\sigma_p}{\sigma}\right)^5\right] \cdot \gamma_j^{\exp[-(\sigma-\sigma_p)^2/0.01\sigma^2]} \Psi(\theta)\,, \tag{36}$$

where the peak-enhancement parameter, γ_J, permits one to vary the spectrum shape from the Pierson-Moskowitz type, $S_{PM}(\sigma)$ (for $\gamma_J = 1$), to the typical JONSWAP type, $S_J(\sigma)$

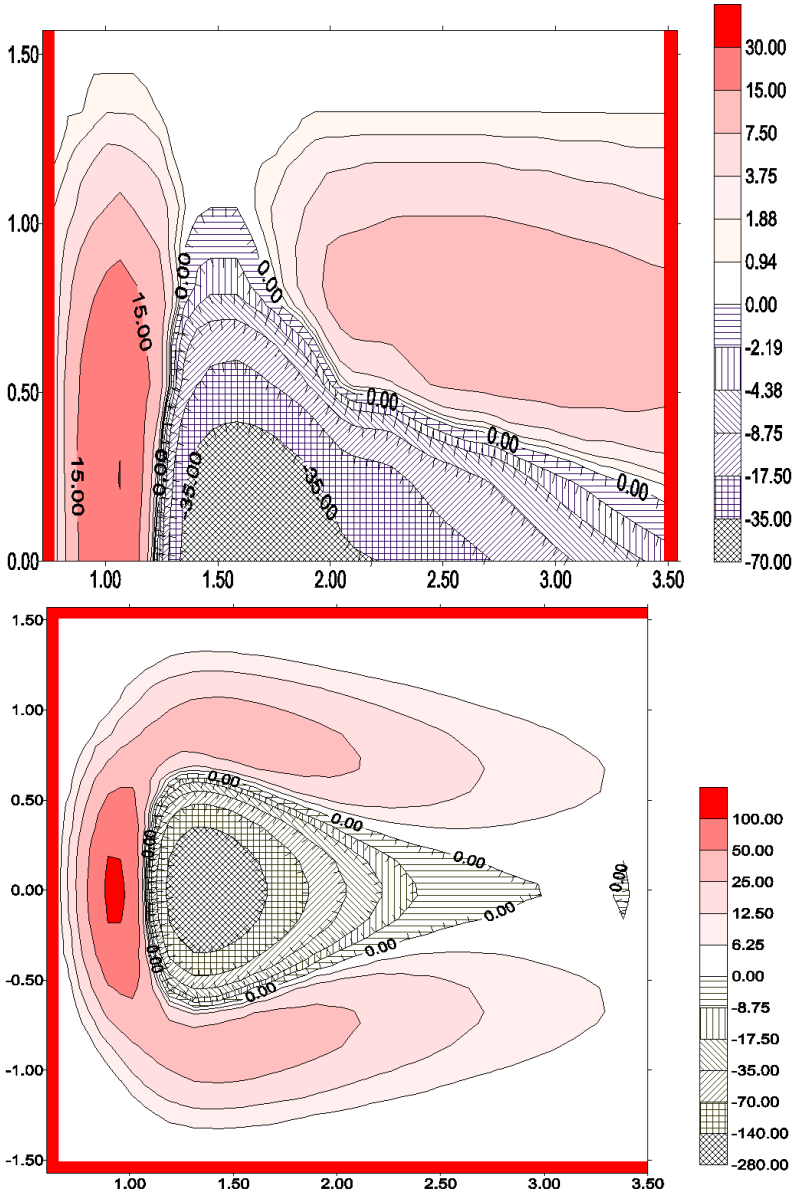

Fig. 1. Two-dimensional nonlinear energy transfers, $T(\sigma,\theta)$, for the spectrum $S_{PM}(\sigma)\cos^2\theta$. **(a)** A numerical calculation of the integral (1) from Polnikov (1989) (the upper half of the (σ,θ) – space only); **(b)** a numerical calculation of the brackets in Eq. (35) (the total (σ,θ) – space). All values are given in units of the dimensional coefficient (31).

(for $\gamma_J = 3.3$), and $\Psi(\theta)$ is the angular spreading function (for example, see paper I). A value of dimensional coefficient C in Eq. (36) is not significant for our calculations, due to normalization of the spectrum by S_p, as it was mentioned above. Calculated values T^+ and T^- for the approximated and exact variants, in units of the constant c given by Eq. (31), are presented in Table 1, for a series of the spectrum shapes.

From Table 1 it follows:

(1) The coefficient c' is not constant, and its value varies depending on the spectrum shape;

(2) The dependence mentioned is enhancing while the derivatives of the spectrum function are increasing;

(3) The united estimation of c' for all types of spectral shapes cannot be found with reasonable accuracy. Thus, it should be determined with a preferable spectral shape. For practical use of the approximation, it is preferable to choose a spectral shape following from the numerical solution of the kinetic equation. The latter is close to the typical JONSWAP spectral shape (see Polnikov (1990)). In such case, the following estimation of the coefficient c' can be accepted:

$$c' = 0.1 \pm 0.05. \qquad (37)$$

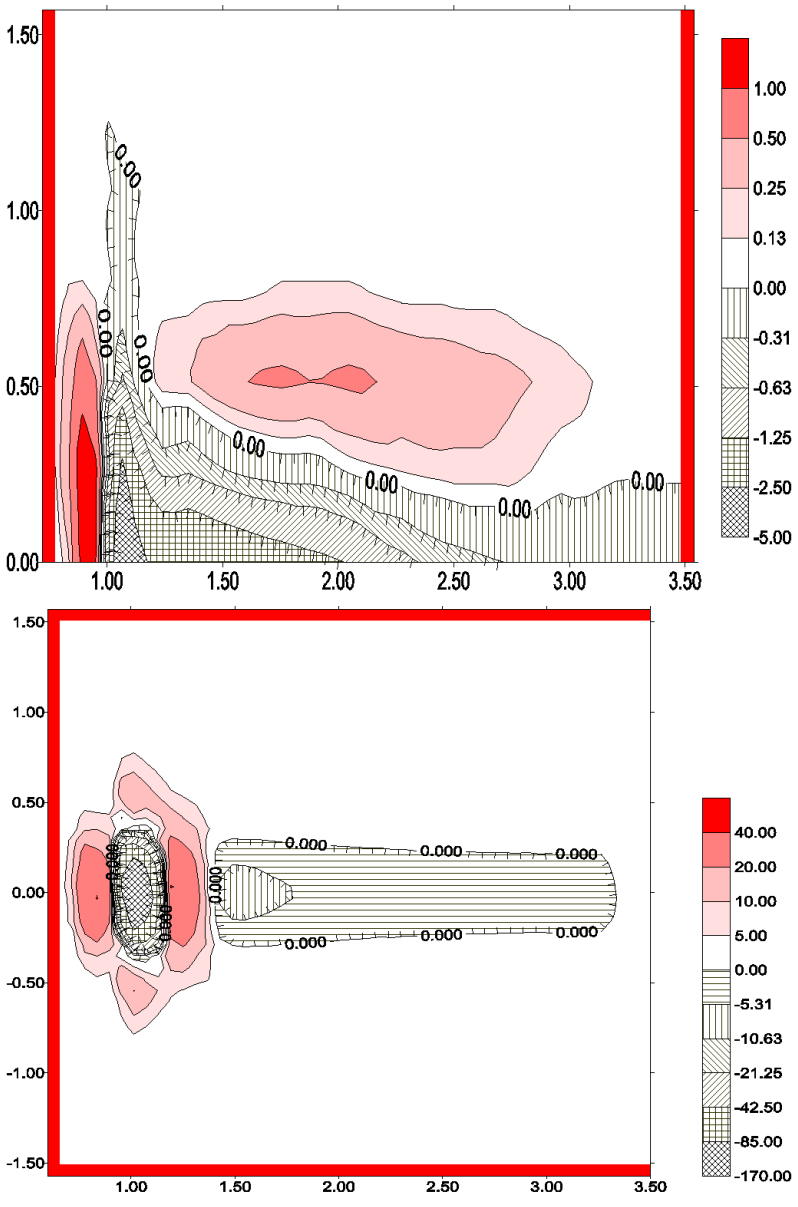

Fig. 2. Two-dimensional nonlinear energy transfers, $T(\sigma,\theta)$, for the spectrum $S_J(\sigma)\cos^8\theta$. See legend in Fig. 1.

An adequacy extent of the estimation obtained can be established by way of comparison of the numerical solutions for Eqs. (1) and (34).

5.3 Analysis of a two-dimensional energy transfer topology

A topology of two-dimensional nonlinear energy transfer for exact and approximate calculations was compared, both for the spectra presented in Table 1 and for the other more complicated spectral shapes (including two-mode and nonsymmetrical shapes). A series of two-dimensional transfers are shown in Figs. 1, 2, and 3.

Analysis of the total set of calculations permits one to draw the following. First, the approximate two-dimensional transfer has a topology close to the exact one. Namely:

– the main extremes, T^+ and T^-, have almost the same values and locations in the (σ,θ)-space;

– local extremes exist as well, but their locations are shifted remarkably;

– for nonsymmetrical spectral shapes, locations and values of the main extremes, T^+ and T^-, are reasonably accurate;

– for two-mode spectra, only locations of the main extremes are correct.

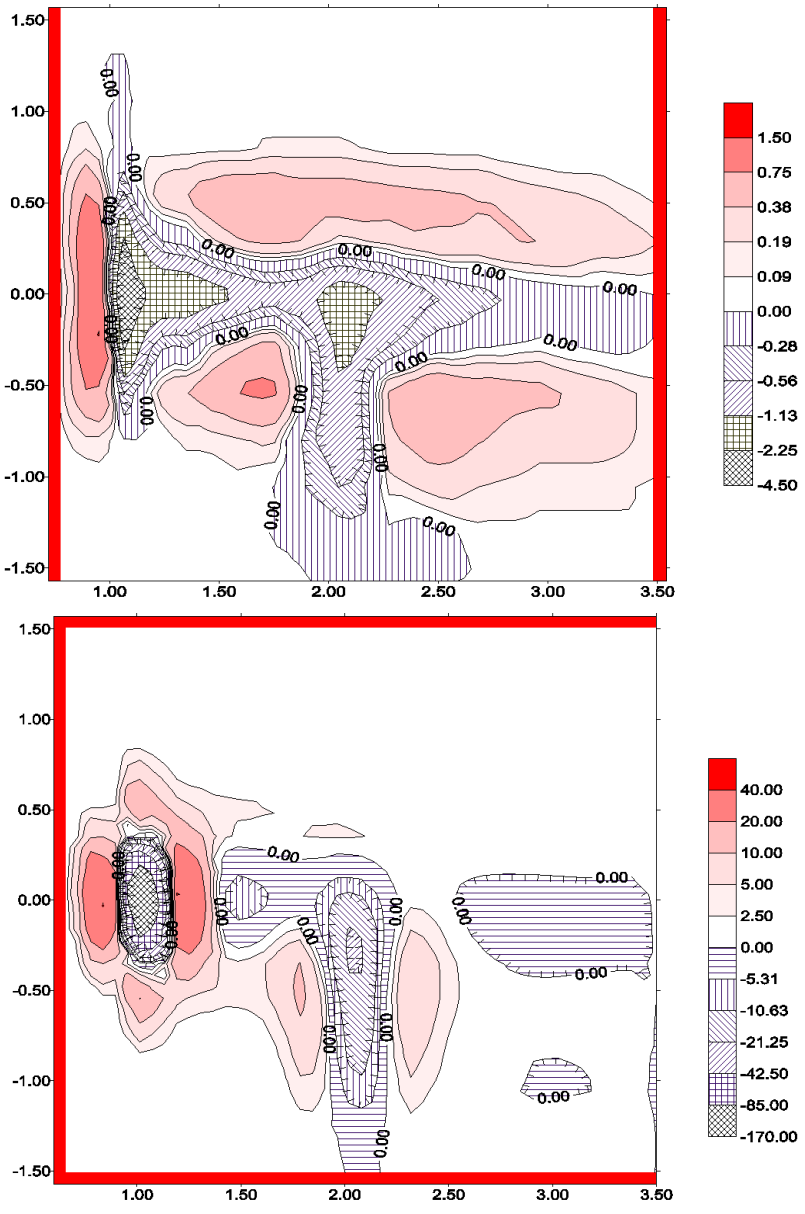

Fig. 3. Two-dimensional nonlinear energy transfers, $T(\sigma, \theta)$, for the two-mode and nonsymmetrical spectrum from Polnikov (1989) of the kind $S_J(\sigma, \sigma_{pl} = 1)\cos^8\theta + 0.8 S_J(\sigma, \sigma_{pl} = 2)\cos^4(\theta + \pi/4)$.

Second, both types of estimations are quantitatively more closer to each other for the spectra with less stronger derivatives in frequency and angle (for example, of the type $S_{PM}(\sigma)\cos^2\theta$). But such type of spectra are not similar to a numerical solution of the kinetic equation, and the estimation of c' of the kind (37) is chosen for the other spectral shapes. Therefore, a mean accuracy of the approximate estimation for the two-dimensional transfer is of the order of 50%. Just this fact is taken into account for the error interval in the estimation (37).

Nevertheless, as a whole, one may state a rather good correspondence of the approximation (35) to the exact transfer, taking into account a lack of any fitting parameters (to say nothing about the coefficient c').

5.4 Numerical solution of Eq. (34)

As it was mentioned above, the final conclusion about diffusion approximation effectiveness can be drawn on the basis of comparison of the numerical solutions for Eqs. (1) and (34). For Eq. (1), the proper solutions are known from Polnikov (1990). Following this paper's investigation logic, let us find numerical solutions for Eq. (34), taking the initial spectra from Table 1.

According to the results obtained by Polnikov (1990), on large time scales of the order of $\tau \geq (10^5 - 10^6)$,

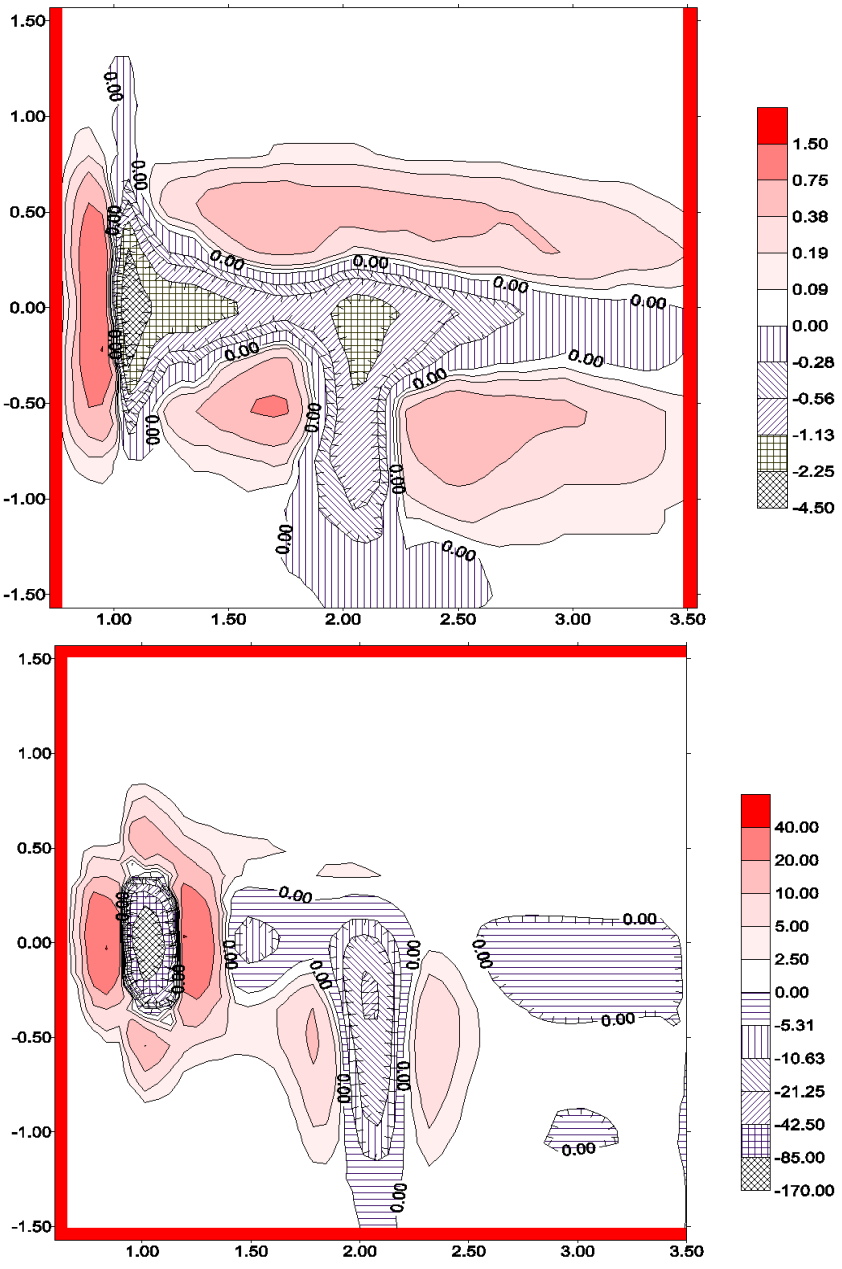

Fig. 4. Two-dimensional nonlinear energy transfers, $T(\sigma, \theta)$, for the two-mode and nonsymmetrical spectrum from Polnikov (1989) of the kind $S_J(\sigma, \sigma_{pl} = 1)\cos^8\theta + 0.8S_J(\sigma, \sigma_{pl} = 2)\cos^4(\theta + \pi/4)$.

$\sigma_p^{-1}(0)$ (σ_p (0)) is the peak frequency of the initial spectrum, usually having the value $\sigma_p(0) = 1$). The spectral shape becomes close to a universal (self-similar) one, depending weakly on an initial spectral shape. Therefore, to estimate the approximation effectiveness, we have to compare some representative spectral parameters for solutions of Eqs. (1) and (34) at the evolution time $t > \tau$.

In Polnikov (1990), the following features for the self-similar spectral shape were revealed:

a) the one-dimensional spectrum, $S(\sigma)$, has a tail fall law of the kind $S(\sigma) \propto \sigma^{-n}$, with the value $n = 7 \pm 1$ at the frequencies $\sigma > \sigma_p$;

b) the frequency width, δ, defined by the relationship

$$\delta = \int S(\sigma)\, d\sigma \,/ S(\sigma_p)\sigma_p, \tag{38}$$

has a small varying value of the order of $\delta = 0.25 \pm 0.03$;

c) the angle narrowness at the peak frequency, D_p, defined with respect to the general wave propagation direction,

Table 2. Estimations of the spectrum parameters at large evolution time scales

Spectrum shape	Initial values		Time of evolution	Final values		
	δ	D	$\tau, \sigma_p^{-1}(0)$	δ	D	n
$S_{PM}(\sigma)\cos^2\theta$	0.67	0.63	1.1×10^6	0.7±0.1	0.9±0.1	4.3±0.1
$S_{PM}(\sigma)\cos^8\theta$	0.67	1.16	2.1×10^6	0.8±0.1	1.3±0.2	4.3±0.1
$S_J(\sigma)\cos^2\theta$	0.33	0.63	2.6×10^6	0.7±0.1	0.9±0.1	4.3±0.1
$S_J(\sigma)\cos^8\theta$	0.33	1.16	3.1×10^6	0.8±0.1	1.3±0.2	4.3±0.1

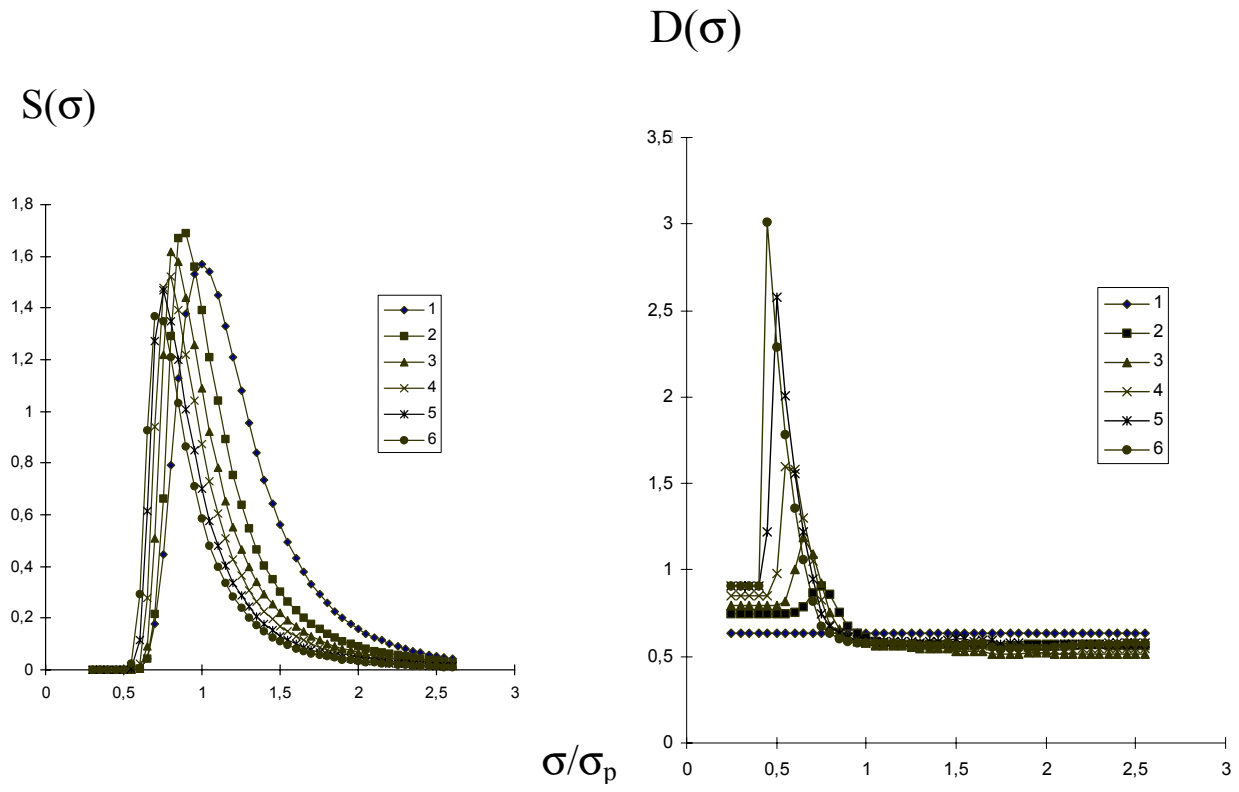

Fig. 5. One-dimensional spectrum shape evolution $S(\sigma, t)$ (panel **(a)**) and angular narrowness function $D(\sigma, t)$ (panel **(b)**) at six moments in time (in units, $\sigma_p^{-1}(0)$): $1-t=0$, $2-t=7.5\times10^3$, $3-t=38\times10^3$, $4-t=111\times10^3$, $5-t=257\times10^3$, $6-t=502\times10^3$. Initial spectrum is $S_{PM}(\sigma)\cos^2\theta$.

θ_p, by the relationship

$$D(\sigma_p) \equiv D = S(\sigma_p, \theta_p) / \int S(\sigma_p, \theta)\, d\theta, \qquad (39)$$

has a small varying value of the order of $D_p = 1.05 \pm 0.05$.

Thus, the criterion of diffusion approximation quality is an extent of the proper parameters' closeness to the values given above at the evolution time $t > \tau \cong (10^5 - 10^6)\sigma_p^{-1}(0)$.

Let us not dwell on technical details of a numerical solution for Eq. (34), but rather let us discuss the results for the mentioned parameters presented in Table 2 for the evolution time $t > \tau \cong 10^6\sigma_p^{-1}(0)$.

As seen from Table 2, the spectral shape parameters obtain more or less constant values which can be estimated as follows

$$n = 4.3 \pm 0.1, \qquad (40)$$

$$\delta = 0.75 \pm 0.2, \qquad (41)$$

$$D_p = 1.1 \pm 0.3. \qquad (42)$$

Thus, a self-similar spectral shape is realized with a certain accuracy for the diffusion approximation (34) as well, though

it differs remarkably from one inherent to the numerical solution for Eq. (1). Details of the difference between these self-similar shapes are rather numerous. As an example, the numerical solution result for Eq. (34), with an initial two-dimensional spectrum of the kind $S_{PM}(\sigma)\cos^2\theta$, is shown in Fig. 4 in terms of the one-dimensional spectrum, $S(\sigma)$, and the angular narrowness function, $D(\sigma)$. But, as one can see, this difference is not so fundamental with respect to the fact of self-similar spectral shape existence.

To our mind, these testing calculations are sufficient for a testing investigation of the diffusion approximation properties.

6 Results analysis and conclusions

Thus, on the basis of a step-by-step analytical integration procedure, we were lucky to find mathematical grounds for the diffusion approximation of the Hasselmann kinetic integral in the form proposed in paper II. Herewith, we did not involve the locality hypothesis for nonlinear interaction among gravity waves. This property follows from the features of resonance surface singularities for the four-wave interactions, the contribution of which, to the integral, is assumed to be representative.

Given the example of the approximation (34), it was shown that this diffusion approximation variant secures correctly a geometry of the two-dimensional transfer, $T(\sigma,\theta)$, not only for simple spectral shapes, $S(\sigma,\theta)$, but also for complicated ones as well (namely two-mode and angular non-symmetric ones). The mean error of the approximated estimation for $T(\sigma,\theta)$ is of the order of 50%, for the total set of the spectra considered. This accuracy level is provided by the principal features of the differential form of the approximation, a quality of which decreases with an increase in spectrum derivatives.

The presented examples of the kinetic equation solutions, found in the diffusion approximation of the kind (34), have shown that the spectrum evolution is similar to one for a numerical solution of Eq. (1), including the fact of self-similar spectral shape existence at large time scales. But the scattering of the self-similar spectral shape parameters is greater, and their values differ from ones for the numerical solutions of Eq. (1). For this approximation the self-similar spectra are more wider in frequency and more narrower in angle, in general.

Taking into account the qualitative likelihood of the two-dimensional spectrum evolution in the diffusion approximation (34), one may draw the conclusion about a possible use of this approximation for practical forecasting of windwaves, when the error of input data is rather great due to its own nature. Herewith, one should take into account that a numerical model fitting, as a rule, diminishes significantly the systematic error created by separate model elements.

In order to elaborate on the diffusion approximation, one could use the possibility to improve the angular (and frequency also) part of the diffusion operator for the nonlinear transfer by way of a better specification for the cubic spectral term under the integral in Eq. (25), taking into account the expansion Eq. (29).

Acknowledgement. The author is thankful to V. Zakharov, V. Krasitskii, and I. Kabatchenko for a stimulating interest to this work. The study was partially supported by the Russian Foundation for Basic Research, project # 99–05–64236.

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