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Properties of Time-Dependent Solutions of the Hasselmann Kinetic Equation

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Abstract—The Hasselmann kinetic equation for deep-water nonlinear gravity waves is studied analytically and numerically to elucidate the existence of a self-similar form of the spectrum resulting from a time-dependent solution of this equation on large time scales. It is shown analytically that, due to the presence of three integrals of motion, the complete self-similarity of solutions to the kinetic equation is impossible. However, this fact does not forbid the existence of an "incomplete self-similarity," which is defined as the establishment of fixed values of the integral parameters of the spectrum in the course of its long-term evolution. This inference is confirmed by the results of a numerical study carried out by invoking both an improved method of calculating the kinetic integral and modern schemes for numerically solving the kinetic equation. The results of numerical experiments are used to determine the limiting characteristics of the form of the energy spectrum of waves on evolutionary scales when the form of the spectrum is completely controlled by nonlinear processes in waves.

1. INTRODUCTION

The problem of describing nonlinear interactions in the wind-wave spectrum was first formulated and studied to a large extent by K. Hasselmann [17–21] and V.E. Zakharov [2–6]. The equation determining the evolution of the wind-wave spectrum in the so-called approximation of a weak wave turbulence [5] is written as the kinetic equation (KE) of the form

$$\frac{\partial N(\mathbf{k})}{\partial t} = \iiint T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \delta(\sigma + \sigma_1 - \sigma_2 - \sigma_3) \times \{N_2N_3(N + N_1) - N_1N(N_2 + N_3)\}d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \quad (1.1)$$

Here, $N_i = N(\mathbf{k}_i)$ is the spectral density of the wave action, $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is the matrix elements of four-wave processes of weakly nonlinear interactions, and $\delta(\mathbf{k})$ and $\delta(\sigma)$ are the Dirac delta functions describing the resonance conditions of interactions of four wave components:

$$\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \quad (1.2a)$$

$$\sigma + \sigma_1 = \sigma_2 + \sigma_3. \quad (1.2b)$$

The most important property of the kinetic equation (1.1) is the conservation of the following three integral quantities in the course of evolution:

the total wave action

$$A = \int N(\mathbf{k})d\mathbf{k}, \quad (1.3a)$$

the total energy

$$E = \int \frac{\sigma N(\mathbf{k})}{4\pi^2 g} d\mathbf{k}, \quad (1.3b)$$

and the total momentum

$$\mathbf{K} = \int \mathbf{k} N(\mathbf{k})d\mathbf{k}. \quad (1.3c)$$

The existence of the integrals of motion for the KE was first proved by Hasselmann back in 1963 [18]. Since that time, studies of the KE properties have primarily focused on the methods of calculating the kinetic integral (KI) [8–10, 12, 20, 22, 24, 25, 27, 28]. A much smaller number of publications were devoted to analytical and numerical studies of the properties of solutions to the KE itself.

For example, in [9, 12, 20, 25, 28], different efficient methods of KI numerical calculations were constructed, and KI properties were examined. The KI properties established in those studies have been checked repeatedly in the course of improving numerical methods of KI calculations and are currently beyond question.

In KE analytical studies, the most significant progress has been made in the works of Zakharov and his coauthors [2–6]. For example, time-independent analytical solutions of the KE of the Kolmogorov type that correspond to a constant energy or action flux over the spectrum were obtained in [3, 5]. It is significant that, in the pioneering work [3], the existence of the Kolmogorov-type solutions was proved only for an infinite frequency range [0, ∞) and only for spectra isotro-

pic in angle. More recently, such solutions were obtained by numerically solving the corresponding equations in a limited frequency range for both isotropic [14] and anisotropic [15] spectra.

Analytical methods to study KE properties were considered further in [4, 7] on the basis of constructing the so-called narrow-directional approximation for the KE. In the recent paper [1], it has been shown that, under certain conditions, the KE can have time-dependent self-similar solutions, and one of them has been obtained analytically. However, an approximate character of the equations used does not enable a complete study of TE properties on the whole.

A relatively small number of studies are devoted to a numerical solution of the KE [8, 13, 22]. In [13], various interesting features were revealed in the evolution of a two-dimensional spectrum, and it has also been found that, on large time scales, the spectral form ceases to depend on the details of the initial spectrum, whereas the integral characteristics of the spectrum vary in a rather narrow range of values. In [13], this result was interpreted as the establishment of a self-similar form of the spectrum on large time scales in the course of a weakly nonlinear evolution of gravity waves, i.e., the evolution of waves described by Eq. (1.1). For a long time, this result escaped the attention of researchers until it was confirmed recently from estimates made in [1] for a number of parameters of a self-similar spectrum. The accuracy of the analytical solution obtained in [1] was tested numerically in [16], which confirmed the result of [1] and explained the distinctions between this result and that obtained in [13].

The problem of existence for time-dependent self-similar solutions to the KE was seemingly solved. However, in the recent paper [22], where testing calculations of the KI were performed, a numerical solution to the KE has been found, and this solution differs from the solutions obtained in [1, 13]. Significant differences lie only in the behavior of the parameters associated with the law by which the tail of the self-similar spectrum decreases, because no other parameters of the form of the calculated spectrum were presented in [22]. Nevertheless, this result forced us to raise the question as to whether the inferences of work [13] are reliable. This question is formulated as follows: Does an actually exact self-similar form of the spectrum of a time-dependent solution to Eq. (1.1) on large time scales exist, and, if so, on what time scales is self-similarity established and what are the parameters of the forms of the corresponding spectra?

The second problem associated with the revision of the inferences of [13] is as follows. In view of the existence of three integrals of motion (1.3a)–(1.3c), it is intuitively clear that the solution of the KE cannot be completely self-similar and independent of the initial conditions on arbitrary time scales. The proof of this statement is contained in the straightforward calculations carried out in the following.

This work is aimed at solving the two problems formulated above. The paper is organized as follows. In Section 2, the evolutionary time scales in question are estimated. An analysis of the results obtained in [1, 13, 16, 22] and specific formulations of the problems are given in Section 3. Section 4 presents analytical calculations that indicate the possible existence of an "incomplete self-similarity" of a solution to the KE. Such a self-similarity manifests itself only in the establishment of fixed values for the integral parameters of the spectrum in the course of its long-term evolution. Section 5 presents a method of numerically studying the problem under consideration. An analysis of results and conclusions are given in Section 6. In the Appendix, an intermediate asymptotic behavior of a self-similar frequency spectrum is proposed.

2. ESTIMATING THE TIME SCALES OF ESTABLISHING A SELF-SIMILAR SOLUTION TO THE KE

Henceforth, a large time scale T is considered to mean the period of spectral evolution in which nonlinear transfer causes the spectrum to completely change its form. This scale can be estimated by dividing the total wave action (1.3a) by the effective magnitude of the action flux F_n, which follows from the conservation law for the total action:

$$\frac{\partial N(\mathbf{k})}{\partial t} + \mathrm{div}_{\mathbf{k}}\mathbf{F}_n = 0, \qquad (2.1)$$

where $\mathbf{F}_n$ is the vector of the action flux. This estimation presents difficulties. However, the value of T can be estimated roughly by taking the ratio of the spectral peak value N_p to the maximum nonlinear transfer $(\partial N/\partial t)_p$, i.e.,

$$T \cong N_p/(\partial N/\partial t)_p. \qquad (2.2)$$

Let us switch in (2.2) from the spectrum of the wave action $N(\mathbf{k})$ to the energy spectrum $S(\mathbf{k})$ and further to the frequency–angle representation for the energy spectrum $S(\sigma, \theta)$ by using the formula

$$N(\mathbf{k})d\mathbf{k} \propto \frac{1}{\sigma}S(\mathbf{k})d\mathbf{k} \propto \frac{g^3}{2\sigma^4}S(\sigma, \theta)d\sigma d\theta. \qquad (2.3)$$

We note that, in (2.3), we used the dispersion relation for waves in deep water $\sigma^2 = gk$, where g is the gravity acceleration. The use of the spectrum $S(\sigma, \theta)$ enables us to easily obtain the required estimate, because the form of the frequency–angle spectrum, its typical values, and the values of the nonlinear transfer for $S(\sigma, \theta)$ are well known [12]. For the typical values $\sigma_p = 1$ rad and $S_p = 0.2$ m^2 s, we have the estimate

$$T \cong 10^5\tau, \qquad (2.4)$$

where $\tau = 2\pi/\sigma_p$ is the period corresponding to the frequency of the spectral peak.

Thus, in the following, we will consider time scales that are no less than an order of magnitude greater than the estimate given in (2.4). It follows that the process of numerically calculating the KI must be repeated a few thousand times to provide the above time of evolution. However, it is well known that the calculation of the collision integral requires much computer time. Therefore, the algorithm of KI computations that is used to solve this problem must be sufficiently fast and must retain the integrals of motion (1.3) with a high accuracy. The most efficient algorithm of this type was designed in [9] using numerical integration methods of the highest accuracy. This algorithm is employed in solving the problem.

3. ANALYSIS OF THE RESULTS OF [1, 13, 16, 22] AND PROBLEM SPECIFICATION

As already noted in the Introduction, the history of studies of numerical solutions to the kinetic equation (1.1) is very short, and the results are rather ambiguous. In order to specify our problem, it is useful to analyze the main results of the four key studies in the order indicated in the heading of this section.

The most important result of [13] lies in the fact that the form of the spectrum $S(\sigma, \theta)$ ceases to depend on the form of the initial spectrum on large time scales. Quantitatively, this fact implies that some integral characteristics of the established spectral form vary in a rather narrow range. In [13], this result was interpreted as the establishment of a self-similar spectral form on large time scales in the course of a weakly nonlinear evolution of gravity waves.

The following characteristic features of the self-similar spectrum were revealed:

(a) a one-dimensional frequency spectrum decreases by the law

$$S(\sigma) \propto \sigma^{-7\pm1} \tag{3.1}$$

at frequencies greater than the frequency of the spectral peak, i.e., at $\sigma > \sigma_p$;

(b) the frequency width of the spectrum B defined as

$$B = \int S(\sigma)d\sigma/S(\sigma_p)\sigma_p \tag{3.2}$$

settles in the range of values $B = 0.25 \pm 0.03$;

(c) the angular width of the spectrum at the frequency of the spectral peak D_p defined with respect to the general direction of wave propagation θ_p as

$$D_p = S(\sigma_p, \theta_p)/S(\sigma_p) \tag{3.3}$$

settles in the range of values $D_p = 1.05 \pm 0.05$.

On the basis of the assumption about the existence of a self-similar spectral form, an analytical estimate was obtained in [13] for the rate at which the frequency of the spectral peak varies with time:

$$\sigma_p \propto t^{-0.1}. \tag{3.4}$$

For a long time, the results of [13] escaped the attention of researchers probably because of difficulties in direct verification of their reliability. Meanwhile, back in 1981, the so-called narrow-directional approximation for the kinetic equation (1.1) was proposed in [4]. The essence of this approximation lies in the expansion of the KI in terms of a specially defined small angular width of the spectrum Δ. As a result, Eq. (1.1) is reduced to a system of two equations for two variables: the one-dimensional spectrum $N(k)$ and the angular width of the spectrum Δ. These equations were studied thoroughly by M.M. Zaslavskii (see references in [1]). Recently, he obtained a self-similar form for $N(k)$ and found an analytical solution to this system of two equations following from Eq. (1.1) [1]. For this purpose, the author of [1] used two independent variables (in addition to the variable g): (1) total action A, which is conserved, and (2) normalized time $t' = tg^{3/2}$. All the variables of the theory (time t, wave number k, and function of the one-dimensional wave action $N(k)$) were redefined through these variables, which enabled the variable t' to be eliminated in the final equations of evolution. In other words, the equations of evolution assumed a self-similar form thus providing a self-similar final solution for the modified spectrum of action.

Without going into details, we only note that, in the theory of [1], the final result depends on the choice of the initial independent variables, whose number must not exceed two. Such a number of variables is sufficient to form the dimensions of all the variables appearing in the theory. The choice of the total action A was due to the fact that, in the narrow-directional approximation, this quantity is conserved exactly, whereas the other two quantities in the set (1.3) are conserved with a certain accuracy (on the order of the small angular width of the spectrum) [1].

The results of the analytical solution found in [1] are as follows:

(1) the law by which the self-similar spectrum decreases corresponds to a one-dimensional frequency spectrum of the form

$$S(\sigma) \propto \sigma^{-13/2}, \tag{3.5}$$

(2) the angular narrowness of the spectrum corresponds to $D_p \cong 1$,

(3) the frequency of the spectral peak varies in time as

$$\sigma_p \cong t^{-1/11}; \tag{3.6}$$

(4) the frequency width of the established spectral form is on the order of $B = 2.7$.

As is seen from a comparison between the results of [13] and [1], their good agreement is observed, except for the value of the parameter B. The same equations were solved numerically in [16], where it was shown that the analytical solution found in [1] is close to the numerical solution in the range around the spectral

peak ($0.5\sigma_p < \sigma < 1.5\sigma_p$) in which the assumption of a small angular width of the spectrum still remains valid. However, in the high-frequency range, $\sigma > 1.5\sigma_p$, the law by which the numerical solution of the self-similar system of two equations decreases assumes the form $N(k) \propto k^{-2}$, which is equivalent to the dependence $S(\sigma) \propto \sigma^{-2}$ in the narrow-directional spectrum [16]. We note that such a significant difference in the frequency dependence of the spectrum between the analytical and numerical solutions is due to the violation of the range of validity of the assumptions accepted in finding the analytical solution (see [1, 16] for details). In addition to the preceding, it was shown in [16] that the difference in estimating B for exact and approximate solutions of the KE is due to the features of the function cutting the spectrum at low frequencies. In principle, this function can be modified without loss in the essence of the theory.

Thus, with regard to the remarks made above, studies [1, 16] have indirectly confirmed the numerical results obtained in [13].

In light of the preceding, work [22], which is devoted to the verification of a new algorithm of calculating the KI, is of interest. It is shown in [22] that, on time scales $T/\tau \cong 10^4$, a numerical solution of Eq. (1.1) for different initial spectra leads to a high-frequency asymptotic behavior ($\sigma > 1.5\sigma_p$) of the form

$$S(\sigma) \propto \sigma^{-4}, \tag{3.7}$$

which is typical for the Kolmogorov spectrum of a constant energy flux toward upper frequencies (see [3]). It is significant that, in [22], the tail of the spectrum of the form (3.7) is implemented in the numerical solution of the KE for a wide variety of the right-hand sides: both without and with allowance for the forcing and dissipation terms on the right-hand side of Eq. (1.1). This fact cannot be explained in the context of the known concepts of the role of the forcing function in forming the spectrum (see [9, 23]). However, such an obvious distinction of the asymptotic behavior (3.7) from the results given in (3.1) and (3.5) stimulates additional studies of the properties of solutions to the KE (1.1).

It is significant that the authors of [22] did not carry out calculations aimed at seeking the spectral form on large time scales; therefore, the above parameters B and D_p were not estimated. Additionally, the scale of time evolution considered in [22] does not even approach the estimate given by (2.4), and the intensity of the spectrum at frequencies $\sigma > 1.5\sigma_p$ is almost an order of magnitude smaller than the intensity of energy-carrying components. It is possible that such small intensity values are beyond the accuracy of the method used for calculations. However, despite the above remarks, a more thorough study of the possible existence of a self-similar spectral form on large time scales is of interest in light of the results obtained in [22]. This study should be carried out by invoking the most accurate methods for solving Eq. (1.1) and calculating the KI.

With regard to the above analysis, the problems considered in this work are formulated as follows:

(1) to analytically substantiate the possible existence of time-dependent steady-state spectral forms for solutions to the KE (1.1) on large time scales,

(2) to numerically solve Eq. (1.1) for a representative series of initial spectra with different initial values of the form parameters B and D_p with the aim to reveal the possible existence of steady-state spectral forms on large time scales of evolution,

(3) to determine the scatter of these parameters and the law by which the steady-state spectral forms decrease and to make an attempt to interpret and parametrize these forms.

In other words, the aim of this work is an exhaustive study of the integral properties of time-dependent numerical solutions to the kinetic equation (1.1).

4. ANALYTICAL APPROACH TO THE PROBLEM

The problem of existence of a time-dependent solution to the KE (1.1) can be solved in several aspects. Let us consider three possible aspects of the problem.

First, from the general considerations presented above in discussing the results of [1], it is evident that the presence of three integrals of motion (1.3) is redundant in constructing a single self-similar equation corresponding to the initial KE (1.1). However, this fact alone does not prove the absence of self-similar solutions.

Second, the absence of exact self-similar solutions to the KE can be proved by contradiction. Indeed, assume that a self-similar solution $S(\sigma, \theta)$ exists. Then, without loss in generality, it can be written as

$$S(\sigma, \theta) = S_p S_A(\tilde{\sigma}) Q_A(\tilde{\sigma}, \theta), \tag{4.1}$$

where $\tilde{\sigma} = \sigma/\sigma_p$ is the normalized frequency, S_p is the peak value of the one-dimensional spectrum, $S_A(\tilde{\sigma})$ is a self-similar frequency function, and $Q_A(\tilde{\sigma}, \theta)$ is a self-similar frequency–angle function that is subject to the normalization condition

$$\int Q_A(\tilde{\sigma}, \theta) d\theta = 1 \tag{4.2}$$

and the condition

$$Q(1, \theta_p) = S_{p2}/S_p, \tag{4.3}$$

where S_{p2} is the peak value of the two-dimensional frequency–angle spectrum. It is significant that the quantities σ_p, S_p, and S_{p2} are functions of the evolution time and can assume arbitrary values.

In this case, the presence of integrals of motion imposes the following limitations on the arbitrary quantities σ_p and S_p:

$$A = \int \frac{\gamma g}{\sigma} S(\sigma, \theta) d\sigma d\theta = \gamma g S_p \int \frac{S_A(\tilde{\sigma})}{\tilde{\sigma}} d\tilde{\sigma} = \gamma g S_p C_a = \text{const}, \quad (4.4)$$

$$E = \int S(\sigma, \theta) d\sigma d\theta = S_p \int S_A(\tilde{\sigma}) d\sigma = S_p \sigma_p \int S_A(\tilde{\sigma}) d\tilde{\sigma} = S_p \sigma_p C_e = \text{const}, \quad (4.5)$$

$$K_x = \int \frac{\gamma g}{\sigma} k \cos(\theta) S(\sigma, \theta) d\sigma d\theta = \gamma S_p \sigma_p^2 \times \int S_A(\tilde{\sigma}) \tilde{\sigma} [\int \cos(\theta) Q_A(\tilde{\sigma}, \theta) d\theta] d\tilde{\sigma} = \gamma S_p \sigma_p^2 \int S_A(\tilde{\sigma}) \tilde{\sigma} F_A(\tilde{\sigma}) d\tilde{\sigma} = \gamma S_p \sigma_p^2 C_k = \text{const}, \quad (4.6)$$

where $\gamma = 4\pi^2$ is the dimensionless coefficient of switching from the action spectrum to the energy spectrum; $F_A(\tilde{\sigma})$ is a new self-similar function; and C_a, C_e, and C_k are some integrals of a self-similar form, which are constant in the case of a complete self-similarity. We note that the integral K_y vanishes automatically in view of an angularly symmetric form of the spectrum.

From the preceding, it is evident, for example, that relations (4.4) and (4.5) cannot be satisfied simultaneously because of a decrease in the peak frequency σ_p of the spectrum in the course of its weakly nonlinear evolution. Specifically, it follows that the conservation conditions for the integral A and E cannot be fulfilled simultaneously, which proves the impossibility of a complete self-similarity for solutions to the KE (1.1).

Third, turning back to the results of [13], one can notice that, for their interpretation, an "incomplete self-similarity" of the spectral form is sufficient. Such an incomplete self-similarity manifests itself in the establishment of fixed integral parameters of the form B and D_p, which depend on the initial spectrum, rather than of a fixed spectral form. Let us show that this result is consistent with the conservation laws (4.4)–(4.6).

We write the expressions for the parameters B and D_p:

$$B = \int S(\sigma, \theta) d\sigma d\theta / S_p \sigma_p = \int S_A(\tilde{\sigma}) d\tilde{\sigma} = C_e, \quad (4.7)$$

$$D_p = \frac{S(\sigma_p, \theta_p)}{S(\sigma_p)} = Q(1, \theta_p) = \frac{S_{p2}}{S_p}. \quad (4.8)$$

It is seen from (4.7) and (4.8) that, in order to ensure the constancy of the parameters B and D_p provided that the integrals A, E, and K_x are conserved simultaneously, it is sufficient to find the conditions of constancy for the energy integral alone (more exactly, for the quantity C_e) without requiring the universality of the functions $S_A(\tilde{\sigma})$ and $Q_A(\tilde{\sigma}, \theta)$. In this case, through the existence of the three independent varying quantities σ_p, S_p, and S_{p2} and of the two nonuniversal functions indicated above, one can simultaneously ensure the existence of the three constants (4.4)–(4.6) and the constancy of the two parameters of the spectral form B and D_p.

Let us describe a possible example of the assumed solution.

First, the peak frequency σ_p decreases slowly with time by the law of the type (3.4) or (3.6). According to (4.5) and (4.7), if the quantity B remains constant, the quantity S_p increases slowly with time and, simultaneously, the total energy remains invariant. The quantity D_p can be conserved without any limitations, because the parameter S_{p2} can assume arbitrary values, and the problem is only to ensure the conservation of the integrals A and K_x.

Let us consider how the process of conserving the total action A can be implemented. For this process, a small decrease in the quantity C_a is required under the conservation of the quantity C_e. This is possible if the form of the function $S_A(\tilde{\sigma})$ has a sufficiently wide peak and a rather long slowly decreasing tail that contains a significant portion of energy and an insignificant portion of wave action. In this case, as a result of a slow sharpening of the peak (through the steepening of the leading edge of the spectrum) and the corresponding rise of the tail with time at a fixed frequency $\tilde{\sigma}$, the two conditions (4.4) and (4.5) can be satisfied.

The process of conserving the quantity K_x is similar. It is implemented through a slow angular widening of the function $Q_A(\tilde{\sigma}, \theta)$ at a fixed frequency $\tilde{\sigma}$, because, according to (4.6), the integral C_k must increase slowly to ensure the constancy of K_x.

It is this result that may be expected in the course of numerically solving Eq. (1.1). In this case, the form parameters B and D_p can depend on their initial values.

5. METHOD AND RESULTS OF THE NUMERICAL SOLUTION OF THE KE (1.1)

The KE is solved numerically using the following method.

First, a representative series of spectra with form parameters B and D_p varying within wide limits are chosen as initial spectra.

Second, a special procedure of numerically solving the KE is developed. This procedure ensures the required accuracy of calculations and the fulfillment of the conservation laws (1.3).

After a large number of testing calculations are performed and the procedures of calculating and repre-

senting the results are debugged, checking calculations are carried out.

The above elements of the method of numerically solving the KE are briefly outlined in what follows.

5.1. Initial Conditions

The weakly nonlinear evolution of the wave spectrum is calculated for the frequency–angle spectrum $S(\sigma, \theta)$ written in the universally accepted form:

$$S(\sigma, \theta) = S(\sigma)Q(\sigma, \theta), \quad (5.1)$$

where $S(\sigma)$ is the frequency spectrum and $Q(\sigma, \theta)$ is the angular distribution of energy.

The frequency function is specified as the JONSWAP spectrum [23]:

$$S(\sigma) = 0.01g^2\sigma^{-5}e^{-\frac{5}{4}\left(\frac{\sigma_p}{\sigma}\right)^4}\gamma^{\exp[-(\sigma-\sigma_p)^2/(2\Delta_J^2\sigma_p^2)]}, \quad (5.2)$$

where

$$\Delta_J = \begin{cases} 0.07 & \text{at} \quad \tilde{\sigma} \le 1 \\ 0.09 & \text{at} \quad \tilde{\sigma} > 1; \end{cases} \quad \text{and} \quad \tilde{\sigma} = \frac{\sigma}{\sigma_p}.$$

For the angular distribution of energy, two approximations are employed. One of them is the conventional cosine distribution of energy

$$Q(\sigma, \theta) = \begin{cases} \left[\dfrac{\Gamma(n_\theta + 1)}{2^{n_\theta}\Gamma^2((n_\theta + 1)/2)}\right]\cos^{n_\theta}(\theta - \bar{\theta}) \\ \text{at} \quad |\theta - \bar{\theta}| \le \pi/2 \\ 0 \quad \text{at} \quad |\theta - \bar{\theta}| \ge \pi/2. \end{cases} \quad (5.3)$$

The second angular distribution is written in the form obtained from the data of the JONSWAP experiment [23]:

$$Q_J(\sigma, \theta) = [2^{2s-1}\pi^{-1}\Gamma(s + 1)/\Gamma(2s + 1)] \times \cos^{2s}((\theta - \bar{\theta})/2), \quad (5.4)$$

where $s = s_{\max}(\tilde{\sigma})^\mu$, $s_{\max} = 9.774$, and $\mu = 4.06$ if $\tilde{\sigma} \le 1$ and $\mu = -2.34$ otherwise.

Table 1. Parameters of the initial spectra

γ	$2s = 2$		$n_\theta = 2$		$n_\theta = 8$	
	B	D_p	B	D_p	B	D_p
1.0	0.69	0.32	0.69	0.64	0.69	1.16
3.3	0.34	0.32	0.34	0.64	0.34	1.16
7.0	0.25	0.32	0.25	0.64	0.25	1.16

Table 1 gives an idea of a representative series of initial spectra and indicates the initial values of the parameters γ, n_θ, and $2s$ and the form parameters B and D_p.

5.2. Algorithm of Solving the Kinetic Equation

The collision integral was calculated numerically by the algorithm described in [9].

The solution of the integro–differential equation (1.1) represents a separate problem. Its complexity lies in obtaining a sufficiently accurate solution at large times $T/\tau \cong 10^7$, when the number of integration steps can reach several tens of thousands. Different numerical methods were used in our calculations: the explicit Euler method of the first order and the Adams method of the second order (predictor–corrector). However, the use of explicit methods turned out to be inefficient. A rather small time step was required because of the instability of the solution at high frequencies. In [13], this difficulty was overcome using an artificial limitation on the allowable amplitude of the solution at high frequencies. However, such an approach leads to a random violation of the conservation condition (1.3), which, in turn, can be responsible for the distortion of the behavior of the solution at large times.

The most efficient numerical scheme turned out to be based on a semiimplicit method [9, 23].

This scheme uses an implicit trapezoid formula:

$$S_{n+1} = S_n + \frac{1}{2}(G(S_n) + G(S_{n+1}))\Delta t. \quad (5.5)$$

In order to obtain the quantity S_{n+1} in an explicit form, it is necessary to solve the equation

$$S_{n+1} - \frac{1}{2}(G(S_{n+1}))\Delta t = S_n + \frac{1}{2}(G(S_n))\Delta t, \quad (5.6)$$

which can be done accurately only in some simplest cases, because the source function G can be a rather complicated function of the spectral density S_{n+1}.

To solve Eq. (1.1) for S_{n+1} in the semiimplicit scheme, the source function G_{n+1} is expanded in a Taylor series:

$$G_{n+1} = G_n + \frac{\partial G_n}{\partial S}\Delta S + \ldots. \quad (5.7)$$

Table 2. Integral characteristics of the established spectral forms on the scales $T/\tau \approx 10^7$

γ	$2s = 2$		$n_\theta = 2$		$n_\theta = 8$	
	B	D_p	B	D_p	B	D_p
1.0	0.358	0.988	0.328	1.31	0.330	1.29
3.3	0.342	0.899	0.320	1.32	0.321	1.12
7.0	0.342	0.963	0.313	1.31	0.326	1.09

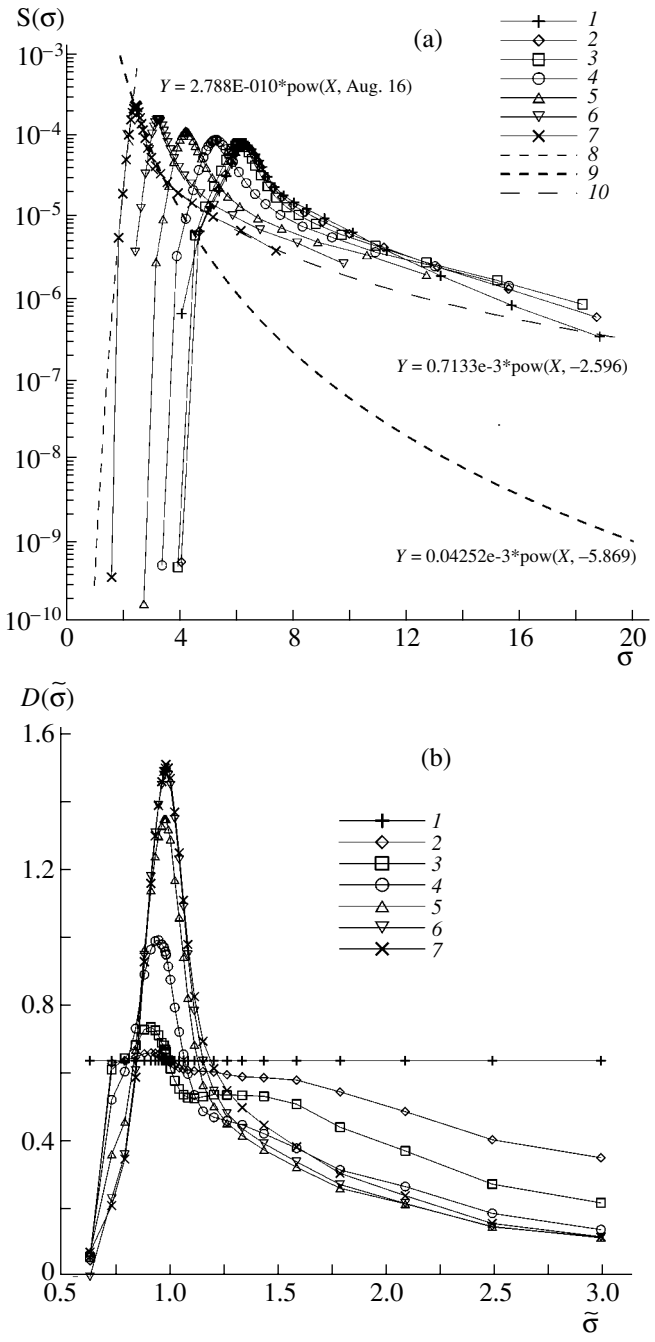

Fig. 1. (a) Time evolution of the frequency spectrum $S(\sigma)$ and (b) the time behavior of the function of angular narrowness $D(\tilde{\sigma})$ for the initial spectrum given by (5.1)–(5.3) at $\gamma = 3.3$ and $n_\theta = 2$. The time is given in the units of $\tau = 2\pi/\sigma_p(0)$. (a) $t = 0.2$ (*1*), 250 (*2*), 1000 (*3*), 10^4 (*4*), 10^5 (*5*), 10^6 (*6*), and 10^7 (*7*); curves *8*, *9*, and *10* correspond to the asymptotic behaviors of the leading and rear edges of the spectral peak and the spectral tail, respectively. (b) $t = 0.2$ (*1*), 250 (*2*), 1000 (*3*), 10^4 (*4*), 10^5 (*5*), 10^6 (*6*), and 10^7 (*7*).

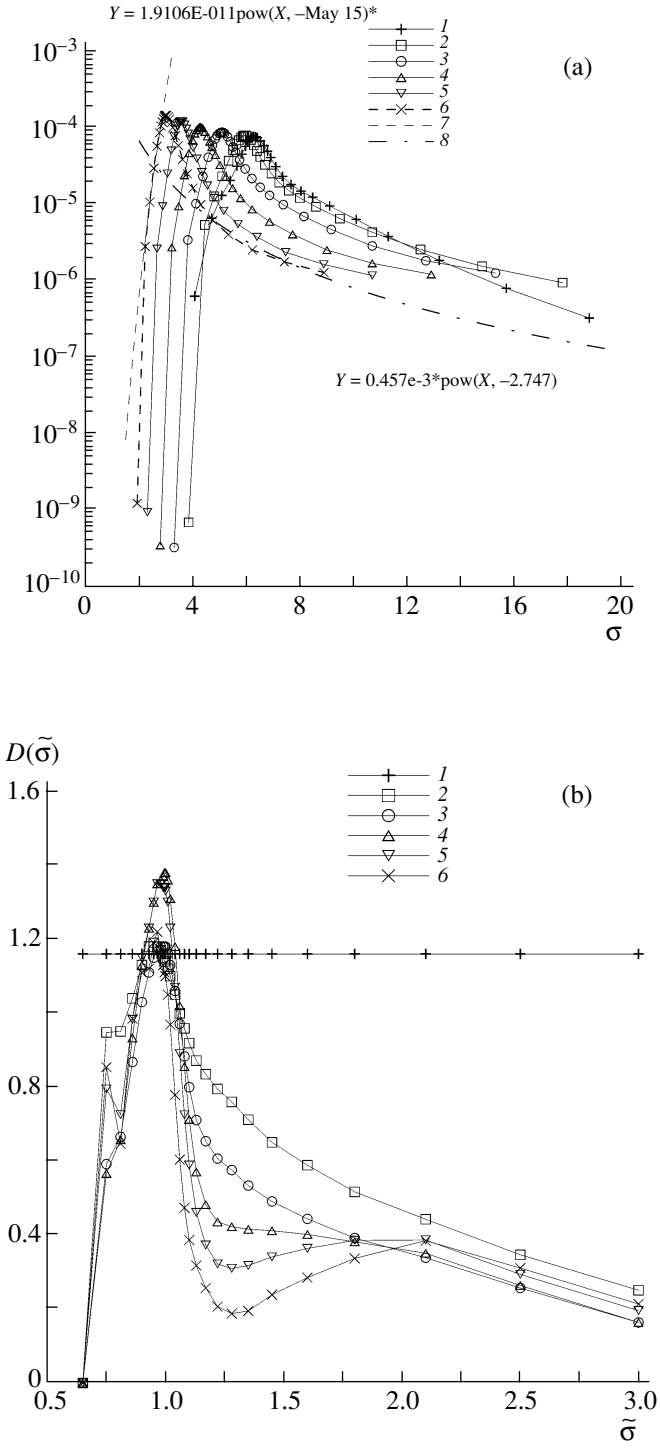

Fig. 2. As in Fig. 1 but for the spectrum (5.1)–(5.3) taken at $\gamma = 3.3$ and $n_\theta = 8$ (the time $t = 250$ and the asymptotic behavior of the rear edge of spectral peak are not shown).

The functional derivative in (5.7) is written as the sum of diagonal (Λ_n) and nondiagonal (N_n) matrices:

$$\frac{\partial G_n}{\partial S} = \Lambda_n + N_n. \qquad (5.8)$$

The substitution of (5.8) into (5.5) yields the following expression:

$$\left[1 - \frac{1}{2}(\Lambda_n + N_n)\Delta t\right]\Delta S = G(S_n)\Delta t, \qquad (5.9)$$

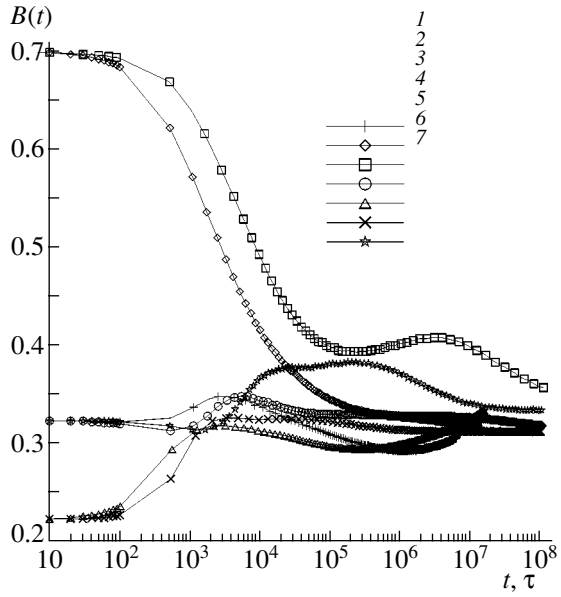

Fig. 3. Time variation of the parameter $B(t)$ for the spectra presented in Table 1: (*1*) $\gamma = 3$ and $n_\theta = 8$, (*2*) $\gamma = 1$ and $n_\theta = 2$, (*3*) $\gamma = 1$ and $n_\theta = 8$, (*4*) $\gamma = 3$ and $n_\theta = 2$, (*5*) $\gamma = 7$ and $n_\theta = 8$, (*6*) $\gamma = 7$ and $n_\theta = 2$, (*7*) $\gamma = 3$ and $n_\theta = 2$. The time is given in the units of $\tau = 2\pi/\sigma_p(0)$.

where $\Delta S = S_{n+1} - S_n$.

As follows from estimations [9, 23], the contribution of the nondiagonal terms in (5.9) is sufficiently small that these terms can be neglected.

Thus, the variation of the spectral density on a time step is

$$\Delta S = G(S_n)\Delta t\left[1 - \frac{1}{2}\Delta t\Lambda_n\right]^{-1}. \qquad (5.10)$$

In further calculations, the approach described above was used, because the numerical solution is fairly stable. It is significant that no additional limitations are imposed here on the magnitude of the numerical solution, as we were forced to do when explicit methods were employed.

It is clear that a numerical error can accumulate at a large number of steps and a stable solution. As a result, the final values turn out to be distorted. An important criterion of the accuracy of a numerical solution is the fulfillment of the conservation laws (1.3). This criterion was constantly tested in our calculations. For example, over the entire course of calculations, the errors in calculating the total energy (1.3b) and the components of the total momentum (1.3c) were not above 10 and 25% (compared to the initial values), respectively.

6. CALCULATION RESULTS AND CONCLUSIONS

The results of calculating the evolution of the frequency–angle wave spectrum have many features, whose analysis requires a separate presentation. However, in this work, we restrict ourselves to an analysis of those integral characteristics of the established spectral forms that were listed in Section 3 in formulating the problem. These characteristics are presented in Table 2 for the initial conditions listed in Table 1. Figures 1–3 illustrate the typical time behaviors of the one-dimensional spectra $S(\sigma)$, the function of angular narrowness $D(\sigma)$, and the parameter of frequency narrowness $B(t)$ for a number of initial spectra.

From a comparison of Tables 1 and 2 and from Figs. 1–3, one can draw the following inferences about the behaviors of the one-dimensional spectrum $S(\tilde{\sigma})$ and the function of angular narrowness $D(\tilde{\sigma})$ on large time scales of evolution:

(1) each of the functions $S(\tilde{\sigma})$ and $D(\tilde{\sigma})$ approaches a certain established form, which is determined by the initial value of the spectrum;

(2) the one-dimensional spectra have the form of a sharpened peak with a very sharp leading edge and a comparatively steep rear edge decreasing by the law $S(\tilde{\sigma}) \propto \tilde{\sigma}^{-6.0 \pm 0.2}$;

(3) in the frequency range $\tilde{\sigma} = \sigma/\sigma_p > 1.5$, the spectrum changes to a smoothly decreasing function $S(\tilde{\sigma}) \propto \tilde{\sigma}^{-2.6 \pm 0.2}$;

(4) the frequency narrowness B, which is determined by the form of the function $S(\tilde{\sigma})$, is stabilized in the range of values 0.32 ± 0.2 and essentially does not depend on the form of the initial spectrum;

(5) the function of angular narrowness $D(\tilde{\sigma})$ has a maximum in the vicinity of the spectral peak and decreases in the ranges of high and low frequencies;

(6) with wave evolution, the range of variations in the parameter D_p narrows, and the established maximum of the parameter D_p lies within the limits $D_p = 1.1 \pm 0.2$ and depends on the initial value of the function of angular narrowness $D_p(0)$.

Thus, despite some discrepancies in numerical values, the results obtained are close to the results of study [13]. Our refinements lie in the fact that, instead of a complete self-similarity, an incomplete self-similarity described in Section 4 actually occurs. Namely, the integral quantities B and D_p and the form of the spectral peak turn out to be fixed. The entire spectrum exhibits two portions: a sharp peak in the frequency range $0.5\sigma_p < \sigma < 1.5\sigma_p$ and a slowly decreasing tail at $\sigma > 1.5\sigma_p$. An example of the parametrization of such a spectrum is presented in the Appendix. As is shown in Section 4, it is this spectral form that ensures the conservation of the three integrals of motion (1.3). Therefore, there is good reason to believe that the result obtained is reliable and final.

In connection with the above inference about the form of the one-dimensional spectrum, we notice that

the analysis of this form made in [13] was restricted to the range $\sigma \le 1.5\sigma_p$, which did not allow the estimation of the law of decreasing the tail of the spectrum. In turn, two portions of the spectrum are also revealed in [22]. However, it is important that the tail of the form $S(\sigma) \propto \sigma^{-4}$, obtained in [22] and typical of the Kolmogorov spectrum, does not appear in our calculations. Such a discrepancy seems to be due to specific details in the algorithms of numerically solving the KE. For the elucidation of the cause of this discrepancy, additional studies and numerical solutions of the KE based on an independent method are necessary. This problem and some other aspects of describing the evolution of the forms of more complicated two-dimensional spectra (bimodal and angularly asymmetric spectra) will be examined in our further studies.

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APPENDIX

Approximation of the Frequency Spectrum

Numerical calculations indicate that, at large times, the spectra exhibit a sort of a self-similar behavior. Therefore, if the law of time variations in the frequency of the spectral peak $\sigma_p = \sigma_p(t)$ is known, the approximation of such a frequency spectrum can be represented (using the dimensions of the variables) in the following form:

$$S(\sigma) = m_0\sigma^{-1}F\left(\frac{\sigma}{\sigma_p}, \frac{A\sigma_p}{m_0}\right). \quad (A.1)$$

The function F can be specified in such a way that the form of the spectrum corresponds to the results of numerical calculations:

$$S(\sigma) = m_0(n+1)\left(\frac{\sigma_p}{\sigma}\right)^n \frac{1}{\sigma}\exp\left(-\frac{n+1}{n}\left(\frac{\sigma_p}{\sigma}\right)^n\right), \quad (A.2)$$

where m_0 is the zero-order moment of the spectrum and n is a dimensionless parameter to be determined. As follows from numerical calculations, its values vary in the limits that can be taken to be $0.5\sigma_p \le \sigma \le 1.5\sigma_p$ for the main energy-carrying range ($n \approx 6.0 \pm 0.2$). We note that the integration of the spectrum (A.1) with respect to frequency yields the zero-order moment of the spectrum m_0, which remains constant under changes in the frequency of the spectral peak σ_p.

For a further refinement of the parameter n, we use the integral wave action A, which can be represented as the –1-order moment of the spectrum (A.2):

$$A = m_{-1} = \int_0^\infty \sigma^{-1}S(\sigma)d\sigma = m_0\sigma_p^{-1}\left[\frac{n+1}{n}\right]^{-1/n}\Gamma\left(1+\frac{1}{n}\right). \quad (A.3)$$

If $n \ge 5$, this expression can be estimated at

$$A \cong m_0\sigma_p^{-1}e^{-C/n}, \quad (A.4)$$

where $C = 0.5772157$ is Euler's constant.

In the course of the evolution of the wave spectrum, if the quantity A holds, the parameter n is a slowly varying function of the frequency of the spectral peak:

$$n \approx -C/\ln(A\sigma_p/m_0). \quad (A.5)$$

Thus, the approximation of the frequency spectrum (A.2) may be considered to describe an intermediate asymptotic behavior of the spectrum, which conserves the total energy of waves and their action. Additionally, the dependence (A.5) shows that the parameter n increases in the course of wave evolution. As the parameter n increases from 4 to 9, the ratio

$$B = m_0/S(\sigma_p)\sigma_p = \exp\left(\frac{n+1}{n}\right)/(n+1) \quad (A.6)$$

varies between 0.70 and 0.30, which is consistent with the results of numerical calculations.

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